# Financial Econometrics

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Lecture 1

# Returns from a financial asset

Let $P_t$ be the price of a financial asset at time period $t$ with $t \in T$. If the asset was bought at time period $s \in T$ with $s < t$ we define:

- The net return from holding a financial asset from time period $s$ to $t$

$$R_{t,s} = \frac{P_t - P_s}{P_s} \text{ for } P_s \neq 0 \text{ with } R_{t,s} \in (-1, \infty) \tag{1}$$

- The gross return from holding a financial asset from time period $s$ to $t$

$$\rho_{t,s} = \frac{P_t}{P_s} \text{ for } P_s \neq 0 \text{ with } \rho_s \in (0, \infty). \tag{2}$$

- The log return from holding a financial asset from time period $s$ to $t$

$$r_{t,s} = \log\left(\frac{P_t}{P_s}\right) \text{ for } P_s \neq 0 \text{ with } r_{t,s} \in (-\infty, \infty). \tag{3}$$

# Returns from a financial asset

Often $T := \mathbb{Z} = \{\cdots, -1, 0, 1, \cdots\}$ and we define

- The net return from holding a financial asset from time period $t-h$ to $t$

$$R_{t,t-h} := R_{t,h} = \frac{P_t - P_{t-h}}{P_{t-h}} \text{ for } P_{t-h} \neq 0 \text{ with } R_{t,h} \in (-1, \infty) \tag{4}$$

- The gross return from holding a financial asset from time period $t-h$ to $t$

$$\rho_{t,t-h} := \rho_{t,h} = \frac{P_t}{P_{t-h}} \text{ for } P_{t-h} \neq 0 \text{ with } \rho_{t,h} \in (0, \infty). \tag{5}$$

- The log return from holding a financial asset from time period $s$ to $t$

$$r_{t,t-h} := r_{t,h} = \log\left(\frac{P_t}{P_{t-h}}\right) \text{ for } P_{t-h} \neq 0 \text{ with } r_{t,h} \in (-\infty, \infty). \tag{6}$$

# Returns from a financial asset

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There are a number of other factors that impact returns:

- Dividends
- Inflation
- Exchange rates
- Taxes

We will mostly abstract from these and focus on "capital gains."
When the holding period $h = 1$ we write

$$R_t,\ \rho_t,\ r_t$$

# Relations between $R_{t,h}$ and $r_{t,h}$

- If net returns are sufficiently "small" they can be well approximated by log-returns.

Let $0 < x < 1$, then $(0, x)$ be an (open) interval and $g(y) = \log(1 + y)$. $g$ has a derivative at each point $y \in (0, x)$, given by $g'(y) = \frac{1}{1+y}$, and is continuous at 0 and $x$. By the Mean Value Theorem, there exists $\bar{x} \in (0, x)$ such that $g(x) - g(0) = \frac{1}{1+\bar{x}}x$. Since $x \neq 0$ and $g(0) = 0$ we write

$$\frac{g(x)}{x} = \frac{1}{1 + \bar{x}} \text{ and } \lim_{x \to 0} \frac{g(x)}{x} = 1, \tag{7}$$

as $\bar{x} \to 0$ as $x \to 0$. Hence, for $R_{t,h}$ sufficiently close to 0,

$$\log(1 + R_{t,h})/R_{t,h} = r_{t,h}/R_{t,h} \text{ will be close to } 1.$$

## Relations between $R_{t,h}$ and $r_{t,h}$

log returns can be viewed as the continuous "compounding" of net returns. Let $s = 0$ and consider $R_{t_F,0}$ where $t_F > 0$. Then

$$R := R_{t_F,0} = \frac{P_{t_F} - P_0}{P_0} \implies P_{t_F} = P_0(1+R). \qquad (8)$$

Now, consider dividing the time interval $[0, t_F]$ into $K$ subintervals of size $t_F/K$. Then, $K$ times compounding gives.

$$P_{t_F} = P_0\left(1 + \frac{R}{K}\right)^K \qquad (9)$$

Using (7) and (8) we get $(1+R) = \left(1+\frac{R}{K}\right)^K$ and taking limits as $K \to \infty$ gives

$$\frac{P_{t_F}}{P_0} = 1 + R = \lim_{K\to\infty}\left(1+\frac{R}{K}\right)^K = \exp(R).$$

Thus, $\log \frac{P_{t_F}}{P_0} = r_{t_F,0} = R$.

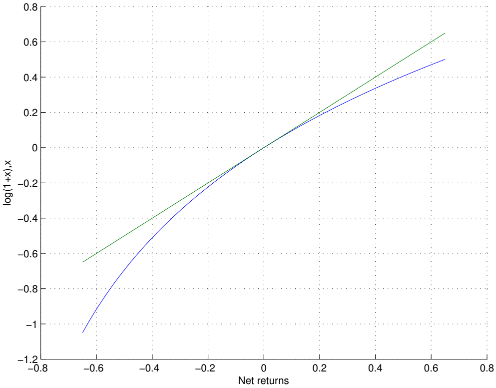


Figure: Graph of log returns and net returns

The Figure shows the graphs of log-returns (blue line) and net returns (green line) against net returns. See the MATLAB code `returns.m` to produce this Figure.

# Additivity of log - returns

log-returns from holding an asset for $h$ periods can be written as the sum of $h$ one period log-returns.

$$\begin{aligned} r_{t,h} = \log P_t - \log P_{t-h} &= \log\left(\frac{P_t}{P_{t-1}}\frac{P_{t-1}}{P_{t-2}}\cdots\frac{P_{t-(h-1)}}{P_{t-h}}\right) \\ &= r_{t,1} + r_{t-1,1} + \cdots + r_{t-(h-1),1} \\ &= r_t + r_{t-1} + \cdots + r_{t-(h-1)} \\ &= \sum_{j=1}^{h} r_{t-(j-1)}. \end{aligned}$$

# Returns on a portfolio

Suppose there are $N \in \mathbb{N}$ assets that compose a portfolio $P$. Assume that $n_i$ shares of asset $i$ are purchased and the price of asset $i$ in time period $t$ is denoted $P_{it}$. The value of the portfolio is

$$V_t = \sum_{i=1}^{n} P_{it} n_i .$$

Let $w_{it} = \frac{P_{it} n_i}{V_t}$ be the share of the investment on asset $i$ in period $t$. Then,

$$\begin{aligned} R_{t+1}^{P} &= \frac{V_{t+1} - V_t}{V_t} = (V_t)^{-1} \left( \sum_{i=1}^{n} P_{i,t+1} n_i - \sum_{i=1}^{n} P_{it} n_i \right) \\ &= (V_t)^{-1} \left( \sum_{i=1}^{n} \frac{P_{i,t+1}}{P_{it}} P_{it} n_i - \sum_{i=1}^{n} P_{it} n_i \right) \\ &= \left( \sum_{i=1}^{n} P_{it} n_i \right)^{-1} \left( \sum_{i=1}^{n} \left( \frac{P_{i,t+1}}{P_{i,t}} - 1 \right) P_{it} n_i \right) \end{aligned}$$

# Returns on a portfolio

$$\begin{aligned} R^P_{t+1} &= \left(\sum_{i=1}^{n} P_{it} n_i\right)^{-1} \left(\sum_{i=1}^{n} R_{i,t+1} P_{it} n_i\right) \\ &= \sum_{i=1}^{n} R_{i,t+1} w_{it} \end{aligned}$$

- The same argument gives

$$\rho^P_{t+1} = \sum_{i=1}^{n} \rho_{i,t+1} w_{it}.$$

- But

$$r^P_{t+1} \neq \sum_{i=1}^{n} r_{i,t+1} w_{it}.$$