# Financial Econometrics

Carlos Brunet Martins-Filho

University of Colorado at Boulder

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Lecture 10

# Modeling dependence - Stochastic processes

- A stochastic process is a collection of time indexed random variables. $\{X_t(\omega) : t \in T\}$, where $T$ is called the index set.
- When $T$ is countable we speak of a discrete process, and when $T$ is uncountable we speak of a continuous process.
- A sample path is a particular realization of the stochastic process through time
- An IID sequence of random variables $\{X_t\}_{t=1,2,\dots}$ is a particular example of a stochastic process.

# Stationarity

**Definition:** a) A stochastic process $\{X_t(\omega) : t \in T\}$ is said to be strictly stationary if for all $n \in \mathbb{N}$, $(t_1, \cdots, t_n)$ and $h$ we have that

$$\begin{pmatrix} X_{t_1} \\ \vdots \\ X_{t_n} \end{pmatrix} \text{ and } \begin{pmatrix} X_{t_1+h} \\ \vdots \\ X_{t_n+h} \end{pmatrix}$$

have the same joint distribution.

b) A stochastic process is covariance stationary if for all $t_1, t_2$ and $h$ we have $E(X_{t_1}) = E(X_{t_2})$ and $cov(X_{t_1}, X_{t_2}) = cov(X_{t_1+h}, X_{t_2+h})$.

# Remarks on stationarity

- Let $t_2 > t_1$ and set $t_2 = t_1 + h$. Then,

$$cov(X_{t_1}, X_{t_2}) = cov(X_{t_2+h}, X_{t_2}) \text{ depends only on } h.$$

- If $t_1 = t_2$ then $V(X_{t_1}) = V(X_{t_1+h})$ for all $h$.
- For a covariance stationary process, $E(X_t) = \mu$ for all $t$, $V(X_t) = \sigma^2$ for all $t$ and

$$cov(X_t, X_\tau) = C(X_0, X_{|t-\tau|}) = \gamma(|t-\tau|)$$

  where $\gamma(h)$ is called the autocovariance function.
- If $T = \{\cdots, -1, 0, 1, \cdots\}$ then $h = 0, \pm 1, \pm 2, \cdots$. Note that $\gamma(0) = V(X_t)$ and we can define an autocorrelation function as

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} \text{ for } h = 0, \pm 1, \pm 2, \cdots.$$

# Example - White noise process

- A weakly stationary process with $\gamma(h) = 0$ for all $h = \pm 1, \pm 2, \cdots$ is called a (weak) white noise.
- If a process is weakly stationary with $\gamma(h) = 0$ for all $h = \pm 1, \pm 2, \cdots$ and each component is normally distributed than we call the process "Gaussian white noise."
- If $\{X_t\}_{t \in \{0, \pm 1, \pm 2, \cdots\}}$ is a stochastic process with $E(X_t | X_{t-1}) = \mu$ for all $t$, then by the LIE

  $$E(X_t) = \mu \text{ for all } t.$$

  Furthermore, $E(X_t X_{t-1}) = E(E(X_t X_{t-1} | X_{t-1})) = E(X_{t-1} E(X_t | X_{t-1})) = \mu^2$ and $\gamma(1) = 0$.
- In fact, for $h = \pm 1, \pm 2, \cdots$ we have $\gamma(h) = 0$.

# Autoregressive processes

- We start by defining the process $X_t$ as,

$$X_t = \phi_0 + \phi_1 X_{t-1} + \varepsilon_t, \; t = 1, 2, \cdots \tag{1}$$

where $\phi_0, \phi_1$ are unknown parameters and $\{\varepsilon_t\}_{t=1,2,,\cdots}$ is a white noise process with $E(\varepsilon_t) = 0$ and $V(\varepsilon_t) = \sigma^2$.

- The process $\{X_t\}$ is called an autoregressive process of order 1 and we write $\{X_t\} = AR(1)$. The order refers to the fact that there is one lag of $X_t$ on the right hand side of (1).
- $\{X_t\} = AR(p)$ where $p$ is a positive integer if

$$X_t = \phi_0 + \phi_1 X_{t-1} + \cdots \phi_p X_{t-p} + \varepsilon_t \; t = 1, 2, \cdots \tag{2}$$

for unknown parameters $\phi_0, \phi_1, \cdots \phi_p$.

# Autoregressive processes

- By repeated substitution (say $m$ times) of the lag term on the right hand side of (1), we obtain

$$X_t = \phi_0(1 + \phi_1 + \phi_1^2 + \cdots + \phi_1^m) + \phi_1^{m+1} X_{t-m-1} + \phi_1^m \varepsilon_{t-m} + \cdots + \phi_1 \varepsilon_{t-1} + \varepsilon_t.$$

- If $|\phi_1| < 1$ then $\lim_{m\to\infty}(1 + \phi_1 + \phi_1^2 + \cdots + \phi_1^m) = \frac{1}{1-\phi_1}$ and $\phi_1^{m+1} \to 0$. Hence, provided that $X_t$ is bounded in a probabilistic sense, then

$$X_t = \frac{\phi_0}{1-\phi_1} + \lim_{m\to\infty} \sum_{j=0}^{m} \phi_1^j \varepsilon_{t-j}. \qquad (3)$$

# Autoregressive processes

- We have
$$E(X_t) = \frac{\phi_0}{1-\phi_1} + E\left(\lim_{m\to\infty}\sum_{j=0}^{m}\phi_1^j\varepsilon_{t-j}\right).$$
.
- It is not always the case that $E(\cdot)$ can be interchanged with a limit operator, but under some regularity conditions (assumed to hold here) this will be possible and we can write
$$E\left(\lim_{m\to\infty}\sum_{j=0}^{m}\phi_1^j\varepsilon_{t-j}\right) = \lim_{m\to\infty}\sum_{j=0}^{m}\phi_1^j E(\varepsilon_{t-j}) = 0$$
given that $\varepsilon_t$ is $WN(0,\sigma^2)$.
- Hence,
$$E(X_t) = \frac{\phi_0}{1-\phi_1} \text{ which does not depend on } t. \qquad (4)$$

# Autoregressive processes

- Using similar arguments we obtain

$$V(X_t) = \frac{1}{1-\phi_1^2}\sigma^2 \text{ which does not depend on } t. \tag{5}$$

and

$$cov(X_t, X_{t-h}) = \frac{\phi_1^{|h|}}{1-\phi_1^2}\sigma^2 \text{ for all } h \text{ and } t. \tag{6}$$

Equations (4), (5) and (6) show that and AR(1) process with $|\phi_1| < 1$ is covariance stationary.

- In addition, from (4), if we define $\mu = \frac{\phi_0}{1-\phi_1}$ then $\phi_0 = \mu - \mu\phi_1$ and

$$\begin{aligned} X_t &= \mu - \mu\phi_1 + \phi_1 X_{t-1} + \varepsilon_t &(7)\\ X_t - \mu &= \phi_1(X_{t-1} - \mu) + \varepsilon_t. &(8) \end{aligned}$$

Note that the equivalence of (1) and (8) depends on $\mu$ existing, i.e., $\phi_1 \neq 1$ which follows from $|\phi_1| < 1$.

# Autoregressive processes

If $\phi_0 = 0$ and $\phi_1 = 1$, then $X_t = X_{t-1} + \varepsilon_t$ and repeated substitution gives

$$X_t = X_{t-m-1} + \varepsilon_{t-m} + \varepsilon_{t-(m-1)} \cdots + \varepsilon_{t-1} + \varepsilon_t. \tag{9}$$

Choosing $m = t - 1$ we have

$$X_t = X_0 + \varepsilon_1 + \varepsilon_2 \cdots + \varepsilon_3 + \varepsilon_t = X_0 + \sum_{j=1}^{t} \varepsilon_j \tag{10}$$

which is a random walk with $E(X_t|X_0) = X_0$ and $V(X_t|X_0) = t\sigma^2$.

# Autoregressive processes

If $|\phi_1| > 1$ (take $\phi_0 = 0$) then

$$X_t = \phi_1^m X_{t-m} + \phi_1^{m-1}\varepsilon_{t-m+1} + \cdots + \phi_1\varepsilon_{t-1} + \varepsilon_t \quad (11)$$

and choosing $m = t$

$$X_t = \phi_1^t X_0 + \phi_1^{t-1}\varepsilon_1 + \cdots + \phi_1\varepsilon_{t-1} + \varepsilon_t \quad (12)$$

with $E(X_t|X_0) = \phi_1^t X_0$ and
$V(X_t|X_0) = \left(\phi_1^{2(t-1)} + \cdots + \phi_1^2 + 1\right)\sigma^2 = \sigma^2 \frac{\phi_1^{2t}-1}{\phi_1^2-1}$ since
$\sum_{i=0}^n r^i = \frac{1-r^{n-1}}{1-r}$ for $n$ finite and $r \neq 0$. Since, $|\phi| > 1$ this variance increases geometrically as $t \to \infty$.