Financial Econometrics

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Lecture 11

Estimating AR(1) parameters - Method of Moments

$E(X_t|X_{t-1}) = \phi_0 + \phi_1 X_{t-1}$ and by the LIE

$$E(X_t) = \phi_0 + \phi_1 E(X_{t-1}).$$

By stationarity $\mu = E(X_t) = E(X_{t-1})$ and consequently

$$E(X_t)(1 - \phi_1) = \phi_0. \tag{1}$$

Also,

$$\begin{aligned} X_t - \mu &= \phi_0 - \mu + \phi_1 X_{t-1} - \phi_1 \mu + \phi_1 \mu + \varepsilon_t \\ &= \phi_0 + \mu(\phi_1 - 1) + \phi_1 (X_{t-1} - \mu) + \varepsilon_t \end{aligned}$$

Thus,

$$E\left((X_t - \mu)(X_{t-1} - \mu)\right) = \phi_1 E\left((X_{t-1} - \mu)^2\right)$$

and consequently

$$\phi_1 = \frac{E\left((X_t - \mu)(X_{t-1} - \mu)\right)}{E\left((X_{t-1} - \mu)^2\right)}. \tag{2}$$

Estimating AR(1) parameters - Method of Moments

The method of moments estimator for ϕ_1 is

$$\hat{\phi}_1 = \frac{n^{-1}\sum_{t=2}^{n}(X_t - \bar{X})(X_{t-1} - \bar{X})}{n^{-1}\sum_{t=1}^{n}(X_t - \bar{X})^2} = \frac{\hat{\gamma}(1)}{\hat{\gamma}(0)}$$

and the method of moments estimator for ϕ_0 is given by

$$\hat{\phi}_0 = \bar{X}(1 - \hat{\phi}_1).$$

Now, note that

$$\begin{aligned} E\left((X_t - E(X_t|X_{t-1}))^2|X_{t-1}\right) &= E\left((X_t - \phi_0 - \phi_1 X_{t-1})^2|X_{t-1}\right) \\ &= E(\varepsilon_t^2|X_{t-1}) \end{aligned}$$

By the LIE

$$E\left((X_t - \phi_0 - \phi_1 X_{t-1})^2\right) = E(\varepsilon_t^2) = \sigma^2.$$

By the method moments, and given $\hat{\phi}_0$ and $\hat{\phi}_1$ we define

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{t=2}^{n}(X_t - \hat{\phi}_0 - \hat{\phi}_1 X_{t-1})^2.$$

Maximum Likelihood Estimation of AR(1)

Since $|\phi_1| < 1$ we have $E(X_t) = \frac{\phi_0}{1-\phi_1}$ and $V(X_t) = \frac{\sigma_\varepsilon^2}{1-\phi_1^2}$.

If $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$, then

$$\varepsilon_t + \phi_0 = X_t - \phi_1 X_{t-1} \sim N(\phi_0, \sigma^2)$$

and we say that $\begin{pmatrix} X_t \\ X_{t-1} \end{pmatrix}$ has a multivariate/bivariate normal distribution, and

$$\begin{pmatrix} X_t \\ X_{t-1} \end{pmatrix} \sim N\left(\begin{pmatrix} \frac{\phi_0}{1-\phi_1} \\ \frac{\phi_0}{1-\phi_1} \end{pmatrix}, \frac{\sigma_\varepsilon^2}{1-\phi_1^2} \begin{pmatrix} 1 & \phi_1 \\ \phi_1 & 1 \end{pmatrix} \right) \text{ and}$$

$$X_t | X_{t-1} \sim N\left(\phi_0 + \phi_1 X_{t-1}, \sigma_\varepsilon^2\right).$$

Consequently, if $\mathbf{X} := \{X_t\}_{t=1}^n$ is a sample, we have

$$\begin{aligned} f_{\mathbf{X}}(x_1, \ldots, x_n) &= f_{X_n|X_{n-1}\cdots X_1}(x_n) f_{X_{n-1}|X_{n-2}\cdots X_1}(x_{n-1}) \cdots f_{X_2|X_1}(x_2) f_{X_1}(x_1) \\ &= f_{X_n|X_{n-1}}(x_n) f_{X_{n-1}|X_{n-2}}(x_{n-1}) \cdots f_{X_2|X_1}(x_2) f_{X_1}(x_1), \end{aligned}$$

Maximum Likelihood Estimation of AR(1)

Under normality,

$$\log f_{\mathbf{X}}(x_1, \ldots, x_n) = \sum_{t=2}^{n} log\ f_{X_t|X_{t-1}}(x_t) + log\ f_{X_1}(x_1)$$

$$= \sum_{t=2}^{n} log\left(\frac{1}{\sqrt{2\pi\sigma_\varepsilon^2}} exp\left(-\frac{1}{2}\left(\frac{X_t - \phi_0 - \phi_1 X_{t-1}}{\sigma_\varepsilon}\right)^2\right)\right)$$

$$+ log\left(\frac{1}{\sqrt{2\pi\frac{\sigma_\varepsilon^2}{1-\phi_1^2}}} exp\left(-\frac{1}{2}\left(\frac{X_1 - \frac{\phi_0}{1-\phi_1}}{\sqrt{\frac{\sigma_\varepsilon^2}{1-\phi_1^2}}}\right)^2\right)\right).$$