Financial Econometrics

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Lecture 12

AR(p) process

For $p \in \mathbb{N}$ let

$$X_t = \phi_0 + \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \varepsilon_t$$

As in the case of AR(1), $E(\varepsilon_t) = 0$, $V(\varepsilon_t) = \sigma^2$, and

$$E(X_t | X_{t-1}, X_{t-2}, \cdots, X_{t-p}) = \phi_0 + \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p}$$

and

$$\begin{aligned} V(X_t | X_{t-1}, X_{t-2}, \cdots, X_{t-p}) &= E\left((X_t - \phi_0 + \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p})^2 \right. \\ &\left. | X_{t-1}, X_{t-2}, \cdots, X_{t-p} \right) \\ &= E\left(\varepsilon_t^2 | X_{t-1}, X_{t-2}, X_{t-p} \right) = \sigma^2 \end{aligned}$$

Estimating AR(p) parameters by least squares

$$\{\tilde{\phi}_0, \tilde{\phi}_1, \cdots, \tilde{\phi}_p\} = \underset{\phi_0,\phi_1,\cdots,\phi_p}{\text{argmin}} \sum_{t=p+1}^{n} (X_t - \phi_0 - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p})^2. \tag{1}$$

Note that if $\mathbf{1}_n = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$, $\mathbf{X} = \begin{pmatrix} X_{p+1} \\ X_{p+2} \\ \vdots \\ X_n \end{pmatrix}$, $\mathbf{X}_{-1} = \begin{pmatrix} X_p \\ X_{p+1} \\ \vdots \\ X_{n-1} \end{pmatrix}$, $\cdots$, $\mathbf{X}_{-p} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{n-p} \end{pmatrix}$ and $\mathbf{R} = \begin{pmatrix} \mathbf{1} & \mathbf{X}_{-1} & \cdots & \mathbf{X}_{-p} \end{pmatrix}$ then we can show that the minimization in (1) is equivalent to

$$\{\tilde{\phi}_0, \tilde{\phi}_1, \cdots, \tilde{\phi}_p\} = \underset{\phi_0,\phi_1,\cdots,\phi_p}{\text{argmin}} (\mathbf{X} - \mathbf{R}\phi)^T (\mathbf{X} - \mathbf{R}\phi). \tag{2}$$

Estimating AR(p) parameters

Taking a first derivative with respect to each ϕ_i $(i = 1, \cdots, p)$, setting to zero and solving for $\tilde{\phi}_i$ we obtain,

$$\tilde{\phi} = \begin{pmatrix} \tilde{\phi}_0 \\ \tilde{\phi}_1 \\ \vdots \\ \tilde{\phi}_p \end{pmatrix} = (\mathbf{R}^T\mathbf{R})^{-1}\mathbf{R}^T\mathbf{X}. \quad (3)$$

The estimator for σ^2 is given by $\tilde{\sigma}^2 = \frac{1}{n}(\mathbf{X} - \mathbf{R}\tilde{\phi})^T(\mathbf{X} - \mathbf{R}\tilde{\phi})$.

Example

Let $r_t = log\ P_t - log\ P_{t-1}$ evolve in accordance to the following stationary AR(3) process

$$r_t = \phi_0 + \phi_1 r_{t-1} + \phi_2 r_{t-2} + \phi_3 r_{t-3} + \varepsilon_t$$

where $\varepsilon_t \sim IID(0, \sigma^2)$. We use the data in `sandp.mat` to estimate the parameters of this process and obtain,

$$\tilde{\phi} = \begin{pmatrix} -0.0000 \\ -0.0874 \\ -0.0814 \\ 0.0320 \end{pmatrix} \text{ and } \tilde{\sigma}^2 = 1.8103 \times 10^{-4}.$$

The estimation (see `ar_3_sandp.m`) indicates that $\frac{\partial E(r_t | r_{t-1}, r_{t-2}, r_{t-3})}{\partial r_{t-1}} < 0$. An AR(1) model

$$r_t = \phi_0 + \phi_1 r_{t-1} + \varepsilon_t$$

is also estimated (see `ar_1_sandp.m`) .

Testing

- We would like to test hypotheses of the type $H_0 : \phi_i = 0$ for $i = 1, \cdots, p$ or more generally, for a known matrix P and vector ρ, we would like to test $H_0 : P\phi = \rho$ against $H_A : P\phi \neq \rho$.
- The basic result needed to conduct this test is the asymptotic normality of $\hat{\phi}$, given by

$$\sqrt{n}(\tilde{\phi} - \phi) \xrightarrow{d} Z \sim N\left(0, \sigma^2 plim_{n\to\infty}\left(\frac{\mathbf{R}'\mathbf{R}}{n}\right)^{-1}\right). \quad (4)$$

Testing

Equation (4) implies that
$\sqrt{n}(P\tilde{\phi} - P\phi) \xrightarrow{d} PZ \sim N\left(0, \sigma^2 P\left(plim_{n\to\infty}\left(\frac{\mathbf{R}'\mathbf{R}}{n}\right)^{-1}\right)P'\right)$,
and under the null hypothesis we have

$$\sqrt{n}(P\tilde{\phi} - \rho) \xrightarrow{d} PZ \sim N\left(0, \sigma^2 P\left(plim_{n\to\infty}\left(\frac{\mathbf{R}'\mathbf{R}}{n}\right)^{-1}\right)P'\right) \quad (5)$$

which can be rewritten as

$$\left(\sigma^2 P\left(plim_{n\to\infty}\left(\frac{\mathbf{R}'\mathbf{R}}{n}\right)^{-1}\right)P'\right)^{-1/2}\sqrt{n}(P\tilde{\phi} - \rho) \xrightarrow{d} N(0, I_v) \quad (6)$$

where v is the number of rows in P and I is an identity matrix.

Testing

The vector on the left hand side of (6) is asymptotically distributed as an independent joint normal. By the definition of a χ^2 distribution, and the fact that for a continuous function g and a sequence of random variables X_n such that $X_n \xrightarrow{d} X$ we have $g(X_n) \xrightarrow{d} g(X)$, we conclude that

$$n(P\tilde{\phi} - \rho)' \left(\sigma^2 P \left(plim_{n\to\infty} \left(\frac{\mathbf{R}'\mathbf{R}}{n} \right)^{-1} \right) P' \right)^{-1} (P\tilde{\phi} - \rho) \xrightarrow{d} \chi^2_v. \tag{7}$$

The use of (7) to test H_0 depends on obtaining a consistent estimator for σ^2. It can be shown that $\tilde{\sigma}^2 \xrightarrow{p} \sigma^2$, hence

$$n(P\tilde{\phi} - \rho)' \left(\tilde{\sigma}^2 P \left(plim_{n\to\infty} \left(\frac{\mathbf{R}'\mathbf{R}}{n} \right)^{-1} \right) P' \right)^{-1} (P\tilde{\phi} - \rho) \xrightarrow{d} \chi^2_v. \tag{8}$$