Financial Econometrics

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Lecture 13

ARCH Models

- The assumption of constant variance implied by stationarity may be inadequate.
- Consider a vector $\begin{pmatrix} Y_t & X_{t1} & \cdots & X_{tp} \end{pmatrix}^T \in \mathbb{R}^{p+1}$, $p \in \mathbb{N}$ and suppose,

$$Y_t = m(X_{t1}, \cdots, X_{tp}) + \varepsilon_t \text{ for } t = 1, 2, \cdots,$$

and note that

$$E(Y_t|X_{t1}, \cdots, X_{tp}) = m(X_{t1}, \cdots, X_{tp}) + E(\varepsilon_t|X_{t1}, \cdots, X_{tp}).$$

If $E(\varepsilon_t|X_{t1}, \cdots, X_{tp}) = 0$, then

$$E(Y_t|X_{t1}, \cdots, X_{tp}) = m(X_{t1}, \cdots, X_{tp})$$

is a regression. Also, if $\sigma^2 = V(\varepsilon_t|X_{t1}, \cdots, X_{tp})$, we have

$$V(Y_t|X_{t1}, \cdots, X_{tp}) = V(\varepsilon_t|X_{t1}, \cdots, X_{tp}) = \sigma^2.$$

ARCH Models

- The last equation is called a "skedastic function."
- A more general formulation is to write,

$$Y_t = m(X_{t1}, \cdots, X_{tp}) + h^{1/2}(X_{t1}, \cdots, X_{tp})\varepsilon_t \text{ for } t = 1, 2, \cdots,$$

for $h : \mathbb{R}^p \to \mathbb{R}$ and $h > 0$. Then,

$$E(Y_t | X_{t1}, \cdots, X_{tp}) = m(X_{t1}, \cdots, X_{tp})$$

and

$$V(Y_t | X_{t1}, \cdots, X_{tp}) = h(X_{t1}, \cdots, X_{tp})$$

if $V(\varepsilon_t | X_{t1}, \cdots, X_{tp}) = 1$.

This is called a conditional location-scale model.

ARCH Models

A special case of this model is the Autoregressive Conditional Heteroskedastic (ARCH) model

- Suppose $\{\varepsilon_t\}_{t\in\mathbb{N}}$ is an IID sequence such that $\varepsilon_t \sim N(0,1)$, and note that $E(\varepsilon_t|\varepsilon_{t-1},\varepsilon_{t-2},\cdots)=0$ and $V(\varepsilon_t|\varepsilon_{t-1},\varepsilon_{t-2},\cdots)=1$.
- A process $\{X_t\}_{t\in\mathbb{N}}$ is said to follow an ARCH(1) model if,

$$X_t = (\alpha_0 + \alpha_1 X_{t-1}^2)^{1/2}\varepsilon_t \text{ for } t = 1,2,\cdots,$$

In the notation of the previous slide, $h(x) = \alpha_0 + \alpha_1 x^2 > 0$.

- It must be that $\alpha_0 + \alpha_1 x^2 > 0$. Since, $h^{(1)}(x) = 2\alpha_1 x$ and $h^{(2)}(x) = 2\alpha_1$, it must be that $\alpha_1 > 0$. Also, h reaches a minimum at $x = 0$ and $h(0) = \alpha_0 > 0$.

ARCH Models

- Then
$$X_t^2 = (\alpha_0 + \alpha_1 X_{t-1}^2)\varepsilon_t^2 \text{ for } t = 1, 2, \cdots ,$$
and $E(X_t^2|X_{t-1}) = \alpha_0 + \alpha_1 X_{t-1}^2$.
- Since $E(X_t|X_{t-1}) = (\alpha_0 + \alpha_1 X_{t-1}^2)^{1/2} E(\varepsilon_t|X_{t-1}) = 0$
$$E(X_t^2|X_{t-1}) = V(X_t|X_{t-1}) = \alpha_0 + \alpha_1 X_{t-1}^2 := h_t$$
- If X_{t-1} is close to its expectation (0) then $h_t \sim \alpha_0$. If not, then $h_t = \alpha_0 + \alpha_1 X_{t-1}^2$.
- If the process is covariance stationary
$$E(E(X_t^2|X_{t-1})) = E(X_t^2) = \alpha_0 + \alpha_1 E(X_{t-1}^2),$$
and $E(X_t^2) = \alpha_0/(1 - \alpha_1)$. This exists if $0 < \alpha_1 < 1$.

ARCH Models

- Note that for any process $\{X_t\}$ such that $E(X_t|X_{t-1}, X_{t-2}, \dots) = \mu$ we have

$$\begin{aligned}\gamma(1) &= Cov(X_t, X_{t-1}) = E(X_t X_{t-1}) - \mu^2 \\ &= E(E(X_t X_{t-1}|X_{t-1}, \cdots)) - \mu^2 \\ &= E(X_{t-1} E(X_t|X_{t-1}, \cdots)) - \mu^2 = \mu E(X_{t-1}) - \mu^2 = 0\end{aligned}$$

- In fact, $\gamma(h) = 0$ for all $h \neq 0$. Hence, ARCH processes are uncorrelated.
- ARCH(1) process is uncorrelated but not independent.
- Contrary to the AR(1) process, that has constant (conditional) variance and changing conditional expectation, ARCH(1) has constant conditional expectation and changing (conditional) variance.

ARCH Models

There is no difficulty in introducing a constant in the ARCH model.

- Let $\mu \in \mathbb{R}$, then

  $$X_t = \mu + (\alpha_0 + \alpha_1(X_{t-1} - \mu)^2)^{1/2}\varepsilon_t \text{ for } t = 1, 2, \cdots ,.$$

  Then, by the LIE, $E(X_t) = \mu$.
- Also,

  $$E((X_t - \mu)^2 | X_{t-1}) = V(X_t | X_{t-1}) = \alpha_0 + \alpha_1(X_{t-1} - \mu)^2$$

  and

  $$V(X_t) = \alpha_0 + \alpha_1 V(X_{t-1})$$

  where $V(X_t) = \alpha_0/(1 - \alpha_1)$.

Example: Combining AR(1) and ARCH(1) Models

- Let $\{X_t\}$ be an ARCH(1) with $\mu = 0$ process and write

$$Y_t - \mu_Y = \phi(Y_{t-1} - \mu_Y) + X_t \text{ for } t = 1, 2, \cdots ,$$

or

$$Y_t - \mu_Y = \phi(Y_{t-1} - \mu_Y) + (\alpha_0 + \alpha_1 X_{t-1}^2)^{1/2} \varepsilon_t \text{ for } t = 1, 2, \cdots ,$$

- Since the ARCH process is uncorrelated Y_t has the autocovariance function of an AR(1) process.
- $E(Y_t | Y_{t-1}) = \mu + \phi Y_{t-1}$ if X_t is independent of Y_t
- $V(Y_t | Y_{t-1}) = \alpha_0 + \alpha_1 X_{t-1}^2$ if X_t is independent of Y_t

ARCH(q) Models

- We say that $\{X_t\}$ follows an ARCH(q) with $\mu = 0$ process if

$$X_t = (\alpha_0 + \alpha_1 X_{t-1}^2 + \alpha_2 X_{t-2}^2 + \cdots + \alpha_q X_{t-q}^2)^{1/2} \varepsilon_t \text{ for } t = 1, 2, \cdots,$$

- For estimation see MATLAB code `garch_11_gau.m`