Financial Econometrics

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Lecture 14

GARCH(1,1) models

- If $\{X_t\}_{t\in\mathbb{N}}$ follows an ARCH(1) model with $E(X_t) = 0$, we have
$$V(X_t|X_{t-1}) := h_t = \alpha_0 + \alpha_1 X_{t-1}^2$$
- $\{X_t\}_{t\in\mathbb{N}}$ is said to follow a GARCH(1,1) model with $E(X_t) = 0$ if
$$X_t = (\alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 h_{t-1})^{1/2}\varepsilon_t$$
where $\varepsilon_t \sim IID(0, \sigma^2)$. Often, it is assumed that $\varepsilon_t \sim NIID(0, \sigma^2)$.

GARCH(1,1) models

- Note that substituting h_{t-1} gives

$$\begin{aligned} X_t &= (\alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1(\alpha_0 + \alpha_1 X_{t-2}^2 + \beta_1 h_{t-2}))^{1/2}\varepsilon_t \\ &= (\alpha_0(1+\beta_1) + \alpha_1(X_{t-1}^2 + \beta_1 X_{t-2}^2) + \beta_1^2 h_{t-2})^{1/2}\varepsilon_t \end{aligned}$$

and substituting h_{t-2} gives

$$\begin{aligned} X_t &= (\alpha_0(1+\beta_1+\beta_1^2) + \alpha_1(X_{t-1}^2 + \beta_1 X_{t-2}^2 + \beta_1^2 X_{t-3}^2) \\ &\quad + \beta_1^3 h_{t-3})^{1/2}\varepsilon_t \end{aligned}$$

After m-substitutions

$$\begin{aligned} X_t = (&\alpha_0(1+\beta_1+\beta_1^2+\cdots+\beta_1^m) + \alpha_1(X_{t-1}^2 + \beta_1 X_{t-2}^2 + \beta_1^2 X_{t-3}^2 \\ &+ \cdots + \beta_1^m X_{t-(m+1)}) + \beta_1^{m+1} h_{t-(m+1)})^{1/2}\varepsilon_t \end{aligned}$$

GARCH(1,1) models

If $0 \leq \beta_1 < 1$ and recalling that $\alpha_0 \geq 0$ and $0 \leq \alpha_1 < 1$, as $m \to \infty$

$$X_t = \left(\frac{\alpha_0}{1-\beta_1} + \alpha_1 (X_{t-1}^2 + \beta_1 X_{t-2}^2 + \beta_1^2 X_{t-3}^2 + \cdots) \right)^{1/2} \varepsilon_t$$

Letting $\delta_0 = \frac{\alpha_0}{1-\beta_1}$, $\delta_1 = \alpha_1 \beta_1^0$, $\delta_2 = \alpha_1 \beta_1$, $\delta_3 = \alpha_1 \beta_1^2 \cdots$

$$X_t = \left(\delta_0 + \sum_{i=1}^{\infty} \delta_i X_{t-i}^2 \right)^{1/2} \varepsilon_t$$

with $\delta_0 \geq 0$, $0 \leq \delta_i \leq 1$. Hence, GARCH(1,1) can be represented as ARCH(∞).

GARCH(1,1) models

- Since $E(\varepsilon_t | X_{t-1}, X_{t-2}, \cdots) = 0$ we have

$$E(X_t | X_{t-1}, X_{t-2}, \cdots) = \left(\delta_0 + \sum_{i=1}^{\infty} \delta_i X_{t-i}^2 \right)^{1/2} \times E(\varepsilon_t | X_{t-1}, X_{t-2}, \cdots) = 0$$

- Since $X_t^2 = \left(\delta_0 + \sum_{i=1}^{\infty} \delta_i X_{t-i}^2 \right) \varepsilon_t^2$ and $E(\varepsilon_t^2 | X_{t-1}, X_{t-2}, \cdots) = \sigma^2$

$$E(X_t^2 | X_{t-1}, X_{t-2}, \cdots) = \left(\delta_0 + \sum_{i=1}^{\infty} \delta_i X_{t-i}^2 \right) \sigma^2$$

- Also, since h_{t-1} depends on $X_{t-2}, X_{t-3,\cdots}$

$$E(X_t^2 | X_{t-1}, X_{t-2}, \cdots) = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 h_{t-1}$$

GARCH(1,1) models

- By the LIE

$$E(E(X_t^2|X_{t-1},X_{t-2},\cdots)) = E(X_t^2) = \left(\delta_0 + \sum_{i=1}^{\infty}\delta_i E(X_{t-i}^2)\right)\sigma^2$$

and we write

$$\gamma(0) = \left(\delta_0 + \sum_{i=1}^{\infty}\delta_i \gamma(0)\right)\sigma^2 \implies \gamma(0) = \frac{\delta_0\sigma^2}{(1-\sigma^2\sum_{i=1}^{\infty}\delta_i)}.$$

Hence,

$$\gamma(0) = \frac{\delta_0\sigma^2}{\left(1-\sigma^2\frac{\alpha_1}{1-\beta_1}\right)}.$$

If $\sigma^2 = 1$, $\gamma(0) = \frac{\alpha_0}{1-\alpha_1-\beta_1}$ where $\alpha_1+\beta_1 < 1$.

- Since, $E(X_t|X_{t-1},X_{t-2},\cdots) = 0$, $\gamma(h) = 0$ for $h = \pm 1, \pm 2, \cdots$

GARCH(q,p) models

- If $\{X_t\}$ is generated by

$$X_t = \left(\alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_q X_{t-q}^2 + \beta_1 h_{t-1} \cdots + \beta_p h_{t-p}\right)^{1/2} \varepsilon_t$$

  we say that X_t follows a GARCH(q,p) model.
- In this case

$$X_t = \left(\frac{\alpha_0}{1-\beta_1-\cdots-\beta_p} + \sum_{i=1}^{\infty} \delta_i X_{t-i}^2\right)^{1/2} \varepsilon_t$$

  where $\delta_i = \alpha_i + \sum_{j=1}^{n} \beta_j \delta_{i-j}$ for $i = 1, 2, \cdots, q$,
  $\delta_i = \sum_{j=1}^{n} \beta_j \delta_{i-j}$ for $i = q+1, \cdots, q$ and $n = \min\{p, i-1\}$.

GARCH(q,p) models

- Note that
$$\gamma(0) = \frac{\alpha_0}{1 - \sum_{i=1}^{p} \beta_i - \sum_{j=1}^{q} \alpha_j}$$
for $1 - \sum_{i=1}^{p} \beta_i - \sum_{j=1}^{q} \alpha_j > 0$
- $\gamma(h) = 0$ for $h = \pm 1, \pm 2, \cdots$
- For estimation see MATLAB codes: `garch_11_gau.m`, `garch_student.m`, `arma_garch_gaussian.m`, `arma_garch_apple.m`