Financial Econometrics

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Lecture 3

Expectation and variance

- If X is continuous

$$E(X) = \int_{-\infty}^{\infty} x f_X(x) dx \text{ and } V(X) = \int_{-\infty}^{\infty} (x - E(X))^2 f_X(x) dx$$

- If X is discrete

$$E(X) = \sum_{i \in \mathbb{N}} x_i p(x_i) \text{ and } V(X) = \sum_{i \in \mathbb{N}} (x_i - E(X))^2 p(x_i)$$

- In both cases $V(X) = E\left((X - E(X))^2\right) = E(X^2) - (E(X))^2$
- $E(X)$ or $V(X)$ may not exist as a real number.

Some functions of random variables

Let X be a random variable, if g is a continuous function defined on the set in which X takes values, then $Y = g(X)$ is a random variable. If g is strictly increasing

$$\begin{aligned} p = P(Y \leq y) = F_Y(y) = P(g(X) \leq y) \\ = P(X \leq g^{-1}(y)) = F_X(g^{-1}(y)), \end{aligned} \tag{1}$$

and differentiating we have

$$\begin{aligned} \frac{d}{dy} F_Y(y) = \frac{d}{dy} \int_{-\infty}^{y} f_Y(z) dz = f_Y(y) \\ = \frac{d}{dy} F_X(g^{-1}(y)) = f_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y). \end{aligned} \tag{2}$$

If g is strictly decreasing, we have

$$\begin{aligned} p = P(Y \leq y) = F_Y(y) = P(g(X) \leq y) \\ = P(X \geq g^{-1}(y)) = 1 - F_X(g^{-1}(y)). \end{aligned} \tag{3}$$

Some functions of random variables

Differentiating we have

$$\frac{d}{dy}F_Y(y) = \frac{d}{dy}\int_{-\infty}^{y} f_Y(z)dz = f_Y(y)$$
$$= \frac{d}{dy}(1 - F_X(g^{-1}(y))) = -f_X(g^{-1}(y))\frac{d}{dy}g^{-1}(y).$$

Hence, for strictly monotone functions we have

$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right|. \tag{4}$$

Note that if $p = F_Y(y)$, from equation (1) we have that

$$F_X^{-1}(p) = g^{-1}(y) \Longrightarrow g(F_X^{-1}(p)) = y = F_Y^{-1}(p). \tag{5}$$

That is, the p-quantile of Y is just the mapping under g of the p-quantile of X.

Example

Let $g(x) = a + bx$ for $b \neq 0$. Then

$$g^{-1}(y) = \frac{y - a}{b}$$

and $\frac{d}{dy} g^{-1}(y) = 1/b$ and $f_Y(y) = f_X\left(\frac{y-a}{b}\right)\frac{1}{b}$.

Also,

$$F_Y^{-1}(p) = a + bF_X^{-1}(p) \text{ if } b > 0$$

and

$$F_Y^{-1}(p) = a + bF_X^{-1}(1 - p) \text{ if } b < 0$$

Independence of random variables

Let $X_1, X_2, \ldots$ be a collection/sequence of random variables. The collection is said to be independent if, and only if, for any finite subset I of $\{1, 2, \ldots\}$ we have

$$P\left(\bigcap_{i\in I}\{X_i \leq x_i\}\right) = \prod_{i\in I} P(X_i \leq x_i).$$

Alternatively, we can write

$$F_I(x_i, i \in I) := P\left(\bigcap_{i\in I}\{X_i \leq x_i\}\right) = \prod_{i\in I} F_{X_i}(x_i).$$

The function $F_I(x_i, i \in I) : \mathbb{R}^{\#I} \to [0, 1]$ is called the joint distribution of $\{X_i : i \in I\}$ and $\#I$ is the number of elements in I (called the cardinality of I).

Independence of random variables

Example 1: Suppose the collection has two random variables, X_1 and X_2. Then, there is only one finite subset of $\{1, 2\}$ to consider, that is, the set itself. Then,

$$P(\{X_1 \leq x_1\} \cap \{X_2 \leq x_2\}) = P(X_1 \leq x_1)P(X_2 \leq x_2).$$

Alternatively, we can write

$$F_I(x_1, x_2) = F_{X_1}(x_1)F_{X_2}(x_2).$$

Independence of random variables

If all $X_1, X_2, \ldots$ are continuous random variables, we can write

$$F_I(x_i, i \in I) := \underbrace{\int_{-\infty}^{x_i} \cdots \int_{-\infty}^{x_i}}_{i \in I} f_{i\in I}(u_{i\in I}) du_{i\in I} = \prod_{i \in I} \int_{-\infty}^{x_i} f_{X_i}(u) du.$$

For example if the collection has only two random variables, we can write

$$F_{(1,2)}(x_1, x_2) := \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f_{(1,2)}(u_1, u_2) du_1 du_2 = \int_{-\infty}^{x_i} f_{X_1}(u) du \times \int_{-\infty}^{x_2} f_{X_2}(u) du.$$

the function $f_{(1,2)}(u_1, u_2)$ is called the joint density of X_1 and X_2.

Marginal and conditional distributions

When dealing with multiple random variables, it is convenient to introduce two new concepts:

- Marginal density functions and marginal distributions
- Conditional density functions and conditional distributions

If $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ is a random vector of continuous random variables, the marginal density of X_1 is given by

$$f_1(x) = \int_{-\infty}^{\infty} f_X(x, u)du \text{ where } f_X \text{ is the joint density of } X.$$

Similarly

$$f_2(x) = \int_{-\infty}^{\infty} f_X(u, x)du$$

Marginal and conditional distributions

The conditional density of X_1 given $X_2 = x_2$

$$f_{1|X_2=x_2}(x) = \frac{f_X(x_1, x_2)}{f_2(x_2)}$$

Similarly

$$f_{2|X_1=x_1}(x) = \frac{f_X(x_1, x_2)}{f_1(x_1)}$$

The marginal, conditional expectations and variances of X_1 are given by

- $E(X_1) = \int u f_1(u) du$, $V(X_1) = \int (u - E(X_1))^2 f_1(u) du$
- $E(X_1|X_2 = x_2) = \int u f_{1|X_2=x_2}(u) du$,
  $V(X_1|X_2 = x_2) = \int (u - E(X_1|X_2 = x_2))^2 f_{1|X_2=x_2}(u) du$

Random sample

- A random sample of size n is a set of observations (values) on a collection $\{X_1, X_2, \cdots, X_n\}$ of independent and identically distributed random variables.
- Here, we need to distinguish the values of the random variables from the random variables themselves. But we represent these two objects by the same notation X_i

The sample average is given by

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

and the sample variance is is given by

$$s_n^2 = \frac{1}{n}\sum_{i=1}^{n} \left(X_i - \bar{X}_n\right)^2 .$$