Financial Econometrics

Professor Martins

University of Colorado at Boulder

Fall 2023

Lecture 4

Covariance

- If X_1, X_2 are random variables, the covariance between X_1 and X_2 is given by

$$\sigma_{X_1,X_2} = E\left((X_1 - E(X_1))(X_2 - E(X_2))\right)$$

If X_1, X_2 are continuous

$$\sigma_{X_1,X_2} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x_1 - E(X_1))(x_2 - E(X_2))f_{(1,2)}(x_1, x_2)dx_1 dx_2$$

- It is easy to show that

$$\sigma_{X_1,X_2} = E(X_1 X_2) - E(X_1)E(X_2).$$

Covariance and indepedence

Suppose X_1 and X_2 are two random variables and g and h are functions. Then,

$$\sigma_{g(X_1)h(X_2)} = E(g(X_1)h(X_2)) - E(g(X_1))E(h(X_2))$$

If X_1 and X_2 are independent

$$E(g(X_1)h(X_2)) = E(g(X_1))E(h(X_2))$$

and $\sigma_{g(X_1)h(X_2)} = 0$. Thus, independence implies zero covariance, but the reverse is not true.

Correlation

- The correlation between X_1 and X_2 is given by

$$\rho_{X_1,X_2} = \frac{\sigma_{X_1,X_2}}{\sqrt{V(X_1)}\sqrt{V(X_2)}}$$

- It is easy to show that $|\rho_{X_1,X_2}| \leq 1$

Example

An important transformation of a normally distributed random variable.
Let $X \sim N(\mu, \sigma^2)$, then $Y = exp(X)$ is said to have a Log-Normal density and we write $Y \sim LN(\mu, \sigma^2)$. Clearly,

$$f_Y(y) = f_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y) = \frac{1}{y} f_X(log\, y) \quad (1)$$

$$= \frac{1}{y} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{(log(y)-\mu)^2}{\sigma^2}} \quad (2)$$

for $0 < y < \infty$. It can be shown that $E(Y) = e^{\left(\mu + \frac{\sigma^2}{2}\right)}$ and $V(Y) = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$.

Example 4

This Figure contains the graphs of 3 log-normal densities (see MATLAB code `lognormgen.m`).

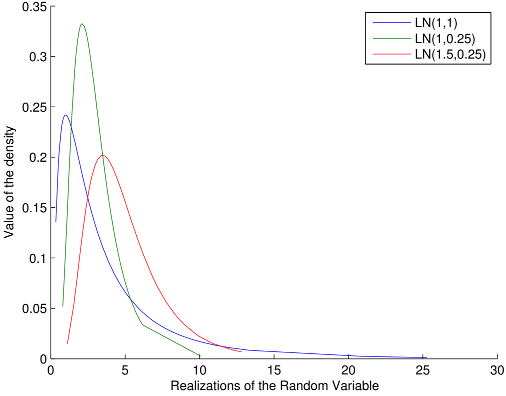

Figure: log-normal densities

Random walk model of returns

Recall that

$$r_{t,h} = \log\left(\frac{P_t}{P_{t-h}}\right) \text{ for } t \in \{\dots, -1, 0, 1, \dots\} \text{ and } h = 1, 2, \dots.$$

and

$$r_{t,h} = \sum_{j=0}^{h-1} r_{t-j,1} = \sum_{j=0}^{h-1} r_{t-j}.$$

Then, if $E(r_{t-j}) = \mu$, $\sigma^2 := V(r_{t-j}) = E(r_{t-j} - \mu)^2$ and $Cov(r_{t-j}, r_{t-i}) = 0$ for all t and $j = 1, 2, \cdots, h$ and $i \neq j$

$$E(r_{t,h}) = h\mu\,, V(r_{t,h}) = h\sigma^2 \text{ and } r_{t,h} \sim N(h\mu, h\sigma^2).$$

Random walk model of returns

We can write

$$r_{t,h} = h\mu + (h\sigma^2)^{1/2} Z \text{ for all } t \text{ where } Z \sim N(0,1)$$

If $F_h(r) = P(r_{t,h} \leq r)$ we have

$$F_h(r) = F_Z\left(\frac{r - h\mu}{(h\sigma^2)^{1/2}}\right)$$

What does this mean about prices?

$$\log P_t = \log P_{t-h} + h\mu + (h\sigma^2)^{1/2} Z \text{ for all } t \text{ where } Z \sim N(0,1)$$

Taking $h = t$ and noting that $r_j \sim N(\mu, \sigma^2)$

$$\log P_t = \log P_0 + t\mu + (t\sigma^2)^{1/2} Z \text{ for all } t \text{ where } Z \sim N(0,1)$$

$$= \log P_0 + r_{t,t} = \log P_0 + \sum_{j=0}^{t-1} r_{t-j} = \log P_0 + \sum_{j=1}^{t} r_j$$

Random walk model of returns

The last equation gives how $\log P_t$ evolves through time. In this case, we say that $\log P_t$ evolves as a "random walk" and P_t is said to evolve as a geometric random walk. If we fix the starting value $\log P_0$

$$E(\log P_t) = \log P_0 + t\mu \text{ and } V(\log P_t) = t\sigma^2.$$

Now, $\exp(\log P_t) = \exp(\log P_0 + t\mu + (t\sigma^2)^{1/2} Z)$

$$P_t = \exp(\log P_0 + r_{t,t}) = \exp(X_t) \text{ where } X_t = \log P_0 + r_{t,t}$$

Since $X_t \sim N(\log P_0 + t\mu, t\sigma^2)$, $P_t \sim LN(\log P_0 + t\mu, t\sigma^2)$ with

$$E(P_t) = \exp\left(\log P_0 + t\mu + \frac{t\sigma^2}{2}\right)$$

$$V(P_t) = \exp\left(2(\log P_0 + t\mu) + t\sigma^2\right)\left(\exp(t\sigma^2) - 1\right)$$