# Financial Econometrics

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Fall 2023

Lecture 5

# Estimation of $\mu$ and $\sigma^2$

We have already estimated $\mu$ and $\sigma^2$ from a random sample $\{r_1, \ldots, r_n\}$ using the method of moments. Now, we study a method called

- Maximum Likelihood Estimation

**Definition.** Let $X \sim f(x;\theta)$ where $\theta \in \Theta \subset \mathbb{R}^p$, $p \in \mathbb{N}$ and the true value of $\theta$ is denoted $\theta_0$. Given a random sample $\{X_i\}_{i=1}^n$, the maximum likelihood estimator of $\theta_0$, whenever it exists, is given by

$$\hat{\theta}_n := \max_{\theta\in\Theta} \log(L_n(\theta; X_1, \cdots, X_n)) = \max_{\theta\in\Theta} \ell_n(\theta; X_1, \cdots, X_n)$$

where $L_n(\theta; X_1, \ldots, X_n) = \prod_{i=1}^n f(X_i;\theta)$.

$L_n$ is called the likelihood function and $\ell_n$ is called the log-likelihood function.

# Estimation of $\mu$ and $\sigma^2$

- If $r_t \sim N(\mu, \sigma^2)$ then,

$$\begin{pmatrix} \hat{\mu}_n \\ \hat{\sigma}_n^2 \end{pmatrix} = \max_{(\mu,\sigma^2)} \sum_{t=1}^{n} \log \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\frac{(r_t-\mu)^2}{\sigma^2}\right)$$

- Note that $\ell_n(\mu, \sigma^2)$ is differentiable with respect to $\mu, \sigma^2$. If a maximum exists, it must satisfy

$$\frac{\partial}{\partial \mu}\ell_n(\hat{\mu}_n, \hat{\sigma}_n^2) = 0 \text{ and } \frac{\partial}{\partial \sigma^2}\ell_n(\hat{\mu}_n, \hat{\sigma}_n^2) = 0$$

- Solving these equations gives

$$\hat{\mu}_n = \frac{1}{n}\sum_{t=1}^{n} r_t \text{ and } \hat{\sigma}_n^2 = \frac{1}{n}\sum_{t=1}^{n}(r_t - \hat{\mu}_n)^2$$

# What are the properties of these estimators?

- $E(\hat{\mu}_n) = \mu$ (unbiased) and $V(\hat{\mu}_n) = \frac{\sigma^2}{n}$ and $\hat{\mu}_n \sim N(\mu, \frac{\sigma^2}{n})$
- Since $E(\hat{\sigma}_n^2) = \left(\frac{n-1}{n}\right)\sigma^2$, we define an alternative unbiased estimator

$$\tilde{\sigma}_n^2 = \frac{1}{n-1}\sum_{t=1}^{n}(r_t - \hat{\mu}_n)^2$$

- More advanced statistics allows us to show that

$$\frac{n-1}{\sigma^2}\tilde{\sigma}_n^2 \sim \chi_{n-1}^2$$

and since if a random variable $X \sim \chi_{n-1}^2$ we have $E(X) = v$ and $V(X) = 2v$, it follows immediately that

$$V(\tilde{\sigma}_n^2) = \frac{2\sigma^4}{n-1}$$

# Confidence interval for $\mu$

- Since $\hat{\mu}_n = \mu + \frac{\sigma}{\sqrt{n}} Z$ where $Z \sim N(0,1)$ it follows immediately that

$$\frac{\hat{\mu}_n - \mu}{\sqrt{\frac{\sigma^2}{n}}} = Z \sim N(0,1)$$

and for any $\alpha \in (0,1)$

$$P\left( z_{(1-\alpha)/2} \leq \frac{\hat{\mu}_n - \mu}{\sqrt{\frac{\sigma^2}{n}}} \leq z_{(1+\alpha)/2} \right) = \alpha$$

where $z_{(1-\alpha)/2}$ is the quantile of order $(1-\alpha)/2$ of $N(0,1)$.

The difficulty with this probability is that $\sigma^2$ must be estimated, since it is unknown.

# Confidence interval for $\mu$

- Consider

$$T = \frac{\hat{\mu}_n - \mu}{\sqrt{\frac{\tilde{\sigma}^2}{n}}} = \frac{\hat{\mu}_n - \mu}{\sqrt{\frac{\tilde{\sigma}^2}{\sigma^2}}\sqrt{\frac{\sigma^2}{n}}}.$$

But $\frac{\hat{\mu}_n-\mu}{\sqrt{\frac{\sigma^2}{n}}} = Z \sim N(0,1)$ and $\frac{\tilde{\sigma}^2}{\sigma^2} = \frac{1}{n-1}X$ where $X \sim \chi^2_{n-1}$.
Hence,

$$T = \frac{\hat{\mu}_n - \mu}{\sqrt{\frac{\tilde{\sigma}^2}{n}}} = \frac{Z}{\sqrt{\frac{\chi^2_{n-1}}{n-1}}} \sim t(n-1).$$

Hence,

$$P\left(T(n-1)_{(1-\alpha)/2} \leq \frac{\hat{\mu}_n - \mu}{\sqrt{\frac{\tilde{\sigma}^2}{n}}} \leq T(n-1)_{(1+\alpha)/2}\right) = \alpha$$

# Confidence interval for $\mu$

- We can, equivalently, write

$$P\left(\hat{\mu}_n + T(n-1)_{(1-\alpha)/2}\sqrt{\frac{\tilde{\sigma}^2}{n}} \leq \mu \leq \hat{\mu}_n \right.$$
$$\left. + T(n-1)_{(1+\alpha)/2}\sqrt{\frac{\tilde{\sigma}^2}{n}}\right) = \alpha$$

# Confidence interval for $\sigma^2$

- Since $\frac{\tilde{\sigma}^2(n-1)}{\sigma^2} \sim \chi^2_{n-1}$, for $\alpha \in (0,1)$

$$P\left(q_{(1-\alpha)/2} \leq \frac{\tilde{\sigma}^2(n-1)}{\sigma^2} \leq q_{(1+\alpha)/2}\right) = \alpha$$

  where $q_{(1-\alpha)/2}$ is the quantile of order $(1-\alpha)/2$ associated with a $\chi^2_{n-1}$. Rearranging, we have

$$P\left(\frac{\tilde{\sigma}^2(n-1)}{q_{(1+\alpha)/2}} \leq \sigma^2 \leq \frac{\tilde{\sigma}^2(n-1)}{q_{(1-\alpha)/2}}\right) = \alpha.$$