Financial Econometrics

Professor Martins

University of Colorado at Boulder

Fall 2023

Lecture 6

Empirical distribution function

- Let $\{X_i\}_{i=1}^n$ be a sequence of independent and identically distributed random variables. The empirical distribution associated with this sequence is

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I_{\{X_i \leq x\}}$$

where $I_{\{X_i \leq x\}}$ is an indicator function. That is, if $S \subseteq \mathbb{X}$ we define $I_S(x) : \mathbb{X} \to \{0, 1\}$ where $I_S(x) = 1$ if $x \in S$ and $I_S(x) = 0$ if $x \notin S$.

- $I_{\{X_i \leq x\}}$ is a discrete random variable taking on 2 values: 0 and 1.

$$E\left(I_{\{X_i \leq x\}}\right) = 1 \times P(\{X_i \leq x\}) + 0 \times P(\{X_i \leq x\}) = F_X(x)$$

$$V\left(I_{\{X_i \leq x\}}\right) = F_X(x)(1 - F_X(x))$$

Empirical distribution function

Hence,

$$E\left(F_n(x)\right) = F_X(x),\ V\left(F_n(x)\right) = \frac{1}{n} F_X(x)(1 - F_X(x))$$

Now, we state Markov's Inequality. If X is a non-negative random variable such that $E(X)$ exist. For any $a > 0$

$$P(X \geq a) \leq \frac{E(X)}{a}.$$

A special case of this inequality is called Chebyshev's Inequality. It is

$$P(|X - E(X)| \geq a) \leq \frac{V(X)}{a^2}.$$

Then, we immediately get

$$P(|F_n(x) - F_X(x)| \geq a) \leq \frac{1}{a^2} \frac{F_X(x)(1 - F_X(x))}{n}.$$

Empirical distribution function

Then,

$$\lim_{n\to\infty} P(|F_n(x) - F_X(x)| \geq a) = 0.$$

We say, in this case, that $F_n(x)$ is a consistent estimator for $F_X(x)$ and write $F_n(x) \xrightarrow{p} F_X(x)$.

In fact, for any estimator θ_n for θ_0, if for any $a > 0$

$$\lim_{n\to\infty} P(|\theta_n - \theta_0| \geq a) = 0.$$

we say that $\theta_n \xrightarrow{p} \theta_0$.

Method of moments estimator for μ and σ^2

- We note that no assumption of normality is needed to obtain the Method of Moment estimators.
- We immediately obtain for any $a > 0$

$$P(|\hat{\mu}_M - \mu| \geq a) \leq \frac{1}{a^2}\frac{\sigma^2}{n},$$

and

$$P(|\tilde{\sigma}^2_M - \sigma^2| \geq a) \leq \frac{1}{a^2}\frac{2\sigma^4}{n-1}.$$

Also, since $\hat{\sigma}^2 = \frac{n-1}{n}\tilde{\sigma}^2_n$

$$P(|\hat{\sigma}^2_n - \sigma^2| \geq a) \leq \frac{1}{a^2}\frac{(n-1)^2}{n^2}\frac{2\sigma^4}{n-1} = \frac{n-1}{n^2}2\sigma^2.$$

Taking limits as $n \to \infty$ shows that $\hat{\mu}_n \xrightarrow{p} \mu$, $\hat{\sigma}^2_n \xrightarrow{p} \sigma^2$ and $\tilde{\sigma}^2_n \xrightarrow{p} \sigma^2$.