Financial Econometrics

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Lecture 8

General properties of MLE

- In many instances the (log) likelihood function is highly nonlinear in the parameters.
- Closed form solutions for the maxima of the likelihood function may not be available.
- Even they are, they may be complex functions of the data and it might not be possible to obtain their distribution and statistical properties.

A more general structure/framework must be developed to obtain the statistical properties of Maximum Likelihood Estimators

Ancillary results

General properties of MLE are based on a collection of approximating results collectively called

"Asymptotic Theory"

There are two fundamental results that we will rely on:

- The Law of Large Numbers (LLN)
- The Central Limit Theorem (CLT)

- Given a random sample $\{X_t\}_{t=1}^{n}$, the LLN gives conditions under which

$$\frac{1}{n}\sum_{t=1}^{n} X_t \text{ converges in probability.}$$

- The CLT gives conditions under which

$$\frac{1}{n}\sum_{t=1}^{n} X_t \text{ converges in distribution.}$$

Convergence in Distribution

Definition: Let $\{X_t\}_{t=1,2,\ldots}$ be a sequence of random variables with distributions $\{F_t\}_{t=1,2,\ldots}$ and X be another random variable with distribution F_X. We say that the sequence $\{X_t\}_{t=1,2,\ldots}$ converges in distribution to X as $n \to \infty$, denoted by

$$X_t \stackrel{d}{\to} X \text{ as } t \to \infty$$

if, and only if,

$$F_t(x) \to F_X(x)$$

for every x that is a point of continuity of F_X as $t \to \infty$.

Kolmogorov's LLN

Theorem: Let $\{X_t\}_{t=1,2,\ldots,n}$ be a sequence of independent and identically distributed random variables with $E(|X_t|) < \infty$ and set $E(X_t) := \mu \in \mathbb{R}$ for all t. Then,

$$\frac{1}{n}\sum_{t=1}^{n} X_t \xrightarrow{p} \mu \text{ as } t \to \infty.$$

- Because the sequence is identically distributed, μ is not indexed by t
- X_t must have an expectation, but need not have a variance or other higher order moments

Lindeberg - Lévy CLT

Theorem: Let $\{X_t\}_{t=1,2,\ldots,n}$ be a sequence of independent and identically distributed random variables with $E(X_t) = \mu \in \mathbb{R}$ and $V(X_t) = \sigma^2 \in (0, \infty)$ for all t. Then,

$$\frac{\frac{1}{n}\sum_{t=1}^{n} X_t - \mu}{\sqrt{V\left(\frac{1}{n}\sum_{t=1}^{n} X_t\right)}} \xrightarrow{d} Z \text{ as } n \to \infty.$$

where $Z \sim N(0,1)$.

Note that

- $\frac{1}{n}\sum_{t=1}^{n} X_t - \mu = \frac{1}{n}\sum_{t=1}^{n}(X_t - \mu)$ and $V\left(\frac{1}{n}\sum_{t=1}^{n} X_t\right) = \frac{\sigma^2}{n}$.

  Hence,
- $\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left(\frac{X_t - \mu}{\sigma}\right) \xrightarrow{d} Z$.