Financial Econometrics

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Lecture 9

Using the LLN and CLT - An important example

Let $\{X_t\}_{t=1,2,\ldots,n}$ be a sequence of IID random variables and

$$F_n(x) = \frac{1}{n}\sum_{t=1}^{n} I_{X_t \leq x}.$$

Note that $E(I_{X_t \leq x}) = F_X(x)$, $V(I_{X_t \leq x}) = (1 - F_X(x))F_X(x)$ and $\{I_{X_t \leq x}\}_{t=1,2,\ldots,n}$ is IID. Hence, by the LLN and the CLT, we have

$$F_n(x) = \frac{1}{n}\sum_{t=1}^{n} I_{X_t \leq x} \xrightarrow{p} E(I_{X_t \leq x}) = F_X(x)$$

and

$$\frac{\frac{1}{n}\sum_{t=1}^{n} I_{X_t \leq x} - E(I_{X_t \leq x})}{\sqrt{\frac{(1-F_X(x))F_X(x)}{n}}} = \frac{\sqrt{n}(F_n(x) - F_x(x))}{\sqrt{(1 - F_X(x))F_X(x)}} \xrightarrow{d} Z \sim N(0,1)$$

or

$$\sqrt{n}(F_n(x) - F_x(x)) \xrightarrow{d} Z \sim N(0, (1 - F_X(x))F_X(x))$$

Using the LLN and CLT - MLE Estimation

Let $\{X_i\}_{i=1}^n$ be a random sample on a random variable $X \sim f_X(x;\theta)$ where $\theta \in \Theta \subset \mathbb{R}$. The ML estimator is defined as

$$\theta_{ML} := \underset{\theta\in\Theta}{\text{argmax}}\ \ell_n(\theta) = \underset{\theta\in\Theta}{\text{argmax}}\ \frac{1}{n}\sum_{i=1}^{n} \log f_X(X_i;\theta). \tag{1}$$

If $\ell_n(\theta)$ has a derivative, i.e., if $\log f_X(X_i;\theta)$ has a derivative, and θ_{ML} is in the interior of Θ, then it must be that θ_{ML} satisfies the following equation

$$\sum_{i=1}^{n} \frac{d}{d\theta} log\ f_X(X_i;\theta_{ML}) = 0. \tag{2}$$

MLE Estimation

If $\frac{d}{d\theta} log\ f_X(X_i;\theta)$ is itself differentiable with respect to θ, by the Mean Value Theorem, there exists $\theta^* = \lambda\theta_{ML} + (1-\lambda)\theta_0$ for $\lambda \in (0,1)$ where θ_0 is the true value of the parameter, such that

$$\begin{aligned}\frac{d}{d\theta} log\ f_X(X_i;\theta_{ML}) &= \frac{d}{d\theta} log\ f_X(X_i;\theta_0) \\ &+ \frac{d^2}{d\theta^2} log\ f_X(X_i;\theta^*)(\theta_{ML}-\theta_0) = 0.\end{aligned}$$

Consequently, we can write

$$-\frac{1}{n}\sum_{i=1}^{n}\frac{d^2}{d\theta^2} log\ f_X(X_i;\theta^*)(\theta_{ML}-\theta_0) = \frac{1}{n}\sum_{i=1}^{n}\frac{d}{d\theta} log\ f_X(X_i;\theta_0). \tag{3}$$

MLE Estimation

Since $\{X_i\}_{i=1,2,\ldots}$ is IID, any sequence $\{g(X_i)\}_{i=1,2,\ldots}$ where g is continuous is also IID. Hence, $\{\frac{d}{d\theta} log\, f_X(X_i;\theta_0)\}_{i=1,2,\ldots}$ is IID and if $E\left(\left|\frac{d}{d\theta} log\, f_X(X_i;\theta_0)\right|\right) < \infty$ then

$$\frac{1}{n}\sum_{i=1}^{n} \frac{d}{d\theta} log\, f_X(X_i;\theta_0) \xrightarrow{p} E\left(\frac{d}{d\theta} log\, f_X(X_i;\theta_0)\right).$$

Now,

$$\begin{aligned} E\left(\frac{d}{d\theta} log\, f_X(X_i;\theta_0)\right) &= \int f_X(u;\theta_0)^{-1} \frac{d}{d\theta} f_X(u;\theta_0) f_X(u;\theta_0) du \\ &= \int \frac{d}{d\theta} f_X(u;\theta_0) du. \end{aligned}$$

If there exists $g(x)$ such that $\left|\frac{d}{d\theta} f_X(x;\theta)\right| \leq g(x)$ and $\int |g(x)| dx < \infty$, then we can interchange integrals and derivatives

$$\int \frac{d}{d\theta} f_X(u;\theta_0) du = \frac{d}{d\theta} \int f_X(u;\theta_0) du = 0$$

MLE Estimation

We conclude that

$$\frac{1}{n}\sum_{i=1}^{n}\frac{d}{d\theta}log\ f_X(X_i;\theta_0)\xrightarrow{p}0.$$

One of g in the previous slide is $g(x)=\sup_{\theta\in\Theta}\left|\frac{d}{d\theta}f_X(x;\theta)\right|$. Similarly, if $E\left(\left|\frac{d^2}{d\theta^2}log\ f_X(X_i;\theta^*)\right|\right)<\infty$, then

$$\frac{1}{n}\sum_{i=1}^{n}\frac{d^2}{d\theta^2}log\ f_X(X_i;\theta^*)\xrightarrow{p}E\left(\frac{d^2}{d\theta^2}log\ f_X(X_i;\theta^*)\right) \tag{4}$$

and provided that $E\left(\frac{d^2}{d\theta^2}log\ f_X(X_i;\theta^*)\right)\neq 0$ for θ^*, we have (from (**??**)) that $\theta_{ML}-\theta_0\xrightarrow{p}0$.

MLE Estimation

Now, multiply equation (**??**) by $\sqrt{n}$, such that we have

$$\sqrt{n}(\theta_{ML} - \theta_0) = \left(-\frac{1}{n} \sum_{i=1}^{n} \frac{d^2}{d\theta^2} log\ f_X(X_i; \theta^*) \right)^{-1} \times \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{d}{d\theta} log\ f_X(X_i; \theta_0) = A_n^{-1} v_n.$$

Since $v_n = \sqrt{n}\frac{1}{n}\sum_{i=1}^{n} \frac{d}{d\theta} log\ f_X(X_i; \theta_0)$ and $E\left(\frac{d}{d\theta} log\ f_X(X_i; \theta_0)\right) = 0$, by the CLT

$$v_n \xrightarrow{d} N\left(0, E\left(\left(\frac{d}{d\theta} log\ f_X(X_i; \theta_0)\right)^2\right)\right)$$

and by the arguments developed above

$$A_n \xrightarrow{p} -E\left(\frac{d^2}{d\theta^2} log\ f_X(X_i; \theta_0)\right) \tag{5}$$

General properties of MLE Estimation

Theorem. (Slutsky Theorem) If f is a continuous function and $\{X_t\}_{t=1,2,\ldots}$ is a sequence of IID random variables, $f(X_t) \xrightarrow{p} f(X)$ provided that $X_t \xrightarrow{p} X$.

We conclude that

$$A_n^{-1} \xrightarrow{p} -\left(E\left(\frac{d^2}{d\theta^2} log\, f_X(X_i;\theta_0)\right)\right)^{-1}.$$

Since $E\left(\frac{d}{d\theta} log\, f_X(X_i;\theta_0)\right) = 0$ we have that $\frac{d}{d\theta} E\left(\frac{d}{d\theta} log\, f_X(X_i;\theta_0)\right) = 0$. Exchanging the expectation and derivatives

$$\frac{d}{d\theta} E\left(\frac{d}{d\theta} log\, f_X(X_i;\theta_0)\right) = \frac{d}{d\theta}\int \frac{d}{d\theta} log\, f_X(u;\theta_0) f_X(u;\theta_0) du$$

$$= \int \left(\frac{d^2}{d\theta^2} log\, f_X(u;\theta_0) f_X(u;\theta_0) + \frac{d}{d\theta} log\, f_X(u;\theta_0) \frac{d}{d\theta} f_X(u;\theta_0)\right) du$$

General properties of MLE Estimation

This implies that

$$-E\left(\frac{d^2}{d\theta^2} log\, f_X(X_i;\theta_0)\right) = E\left(\left(\frac{d}{d\theta} log\, f_X(X_i;\theta_0)\right)^2\right).$$

Theorem. If $\{X_n\}$, $\{Z_n\}$ are sequence of random variables such that $X_n \xrightarrow{p} X$ and $Z_n \xrightarrow{d} Z$, then $X_n Z_n \xrightarrow{d} XZ$.

Hence, we finally conclude

$$\sqrt{n}(\theta_{ML} - \theta_0) \xrightarrow{d} N\left(0, \left(-E\left(\frac{d^2}{d\theta^2} log\, f_X(X_i;\theta_0)\right)\right)^{-1}\right).$$

General properties of MLE Estimation

If $\theta \in \Theta \subset \mathbb{R}^d$ we write

$$\sqrt{n}(\theta_{ML} - \theta_0) \xrightarrow{d} N\left(0, \left(-E\left(\frac{\partial^2}{\partial\theta\partial\theta'} log\, f_X(X_i;\theta_0)\right)\right)^{-1}\right).$$

where $-E\left(\frac{\partial^2}{\partial\theta\partial\theta'} log\, f_X(X_i;\theta_0)\right)$ is a $d \times d$ matrix.

- This matrix is called the *Fisher's Information.*
- For arbitrary estimator $\hat{\theta}$, such that

$$\sqrt{n}(\hat{\theta} - \theta_0) \xrightarrow{d} N(0, V_{\hat{\theta}})$$

  we have $V_{\hat{\theta}} \geq \left(-E\left(\frac{\partial^2}{\partial\theta\partial\theta'} log\, f_X(X_i;\theta_0)\right)\right)^{-1}$.
- The MLE is efficient (minimum variance) among all estimators that are consistent and asymptotically normally distributed. The expression

$$\left(-E\left(\frac{\partial^2}{\partial\theta\partial\theta'} log\, f_X(X_i;\theta_0)\right)\right)^{-1}$$

  is called the Cramér-Rao lower bound.