Math Camp 2025, Professor Martins, Final Examination, Date: 08.15.2025 at 9:00 AM - 11:00 AM.

Instructions: The pages of your set of answers should be numbered consecutively starting at 1. Use only one side of a sheet of paper in answering the question. No writing on the reverse side. You are free to consult notes, books and previous homework sets as you answer the questions below. No communication with other human beings is allowed in crafting your answers, nor can you use the web. Answer all questions.

Question 1: A question about sets, functions, normed spaces and integrals.

1. Show that $(A - B) - C = A - (B \cup C)$ and also that $A - (B - C) = (A - B) \cup (A \cap C)$. Hint: Use De Morgan's Laws.
2. Suppose f and g are one-to-one. Show that $f \circ g$ is one-to-one. Do you have to assume that $f \circ g$ exists?
3. Let $f \in C_{[a,b]}$ for $a < b$ and $a, b \in \mathbb{R}$ and define $I(f) := \int_a^b f(t)dt$. Is I a linear functional on $C_{[a,b]}$? Is it bounded? What is the norm of I if we set $\|f\| := \max_{a \leq x \leq b} |f(x)|$? Hint: You can use properties of Riemann integrals.

Question 2: A question about convergence and linear functions.

1. Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of bounded linear functionals defined on the normed space $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$, i.e., $f_n : \mathbb{X} \to \mathbb{R}$ is bounded and linear. Suppose $\{f_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in the space of bounded linear functions from $\mathbb{X}$ to $\mathbb{R}$, denoted by $B(\mathbb{X}, \mathbb{R})$. Show that if $f(x) = \lim_{n \to \infty} f_n(x)$, $f \in B(\mathbb{X}, \mathbb{R})$.
2. Show that in a Banach space a subset is complete if, and only if, it is closed.

Question 3: A question about matrices.

1. Let A be a symmetric positive definite matrix, show that A^{-1} is symmetric and positive definite.
2. Let B be a matrix be an $m \times n$ matrix. Show that $B^T B$ is positive semi-definite and positive definite if, and only if, the rank of B is n.

Question 4: A question about differentiation.

1. Obtain the partial derivatives and directional derivative (in the direction u) of each of the following functions defined on $\mathbb{R}^n$: a) $f(x) = a^T x$ for $a \in \mathbb{R}^n$ a fixed point; b) $f(x) = x^T L(x)$ where $L \in L(\mathbb{R}^n, \mathbb{R}^n)$.
2. Let $f : \mathbb{X} \to \mathbb{R}$ be Gâteaux differentiable on $\mathbb{X}$. Show that a necessary condition for f to have a local extremum (local minimum or local maximum) at $x_0 \in \mathbb{X}$ is that $\delta_g(x_0; h) = 0$ for all $h \in \mathbb{X}$.