Proof. $A = A \cap \Omega = A \cap (\cup_{i\in\mathbb{N}} E_i) = \cup_{i\in\mathbb{N}}(A \cap E_i)$. The collection $\{(A \cap E_1), (A \cap E_2), \ldots\}$ is pairwise disjoint. Therefore, $P(A) = \sum_{i\in\mathbb{N}} P(A \cap E_i) = \sum_{i\in\mathbb{N}} P(A|E_i)P(E_i)$. ■

Theorem 1.13. *(Bayes' Theorem) Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{E_1, E_2, \ldots\}$ be a partition of Ω. Let $A \in \mathcal{F}$ such that $P(A) \neq 0$. Then,*

$$P(E_i|A) = \frac{P(A|E_i)P(E_i)}{\sum_{j\in\mathbb{N}} P(A|E_j)P(E_j)},$$

Proof. By the previous theorem $P(A) = \sum_{j\in\mathbb{N}} P(A|E_j)P(E_j) \neq 0$. Hence,

$$P(E_i|A) = \frac{P(E_i \cap A)}{P(A)} = \frac{P(A|E_i)P(E_i)}{\sum_{j\in\mathbb{N}} P(A|E_j)P(E_j)}$$

which establishes the desired result. ■

1.3.1 Some remarks on the structure of $\mathbb{R}$ and its Borel sets

In Remark 1.2.1 we defined the Borel σ-algebra associated with a set $\mathbb{X}$ using open sets. We now provide results that are useful in obtaining the Borel sets of $\mathbb{R}$, denoted by $\mathcal{B}(\mathbb{R})$ or $\mathcal{B}_{\mathbb{R}}$. Recall that an open interval on $\mathbb{R}$ is a set $(a, b) := \{x \in \mathbb{R} : a < x < b\}$ and a closed interval is a set $[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$. They are said to be finite if $a, b \in \mathbb{R}$ and infinite if $a = -\infty$ or $b = \infty$.

Definition 1.8. *Let S be an open subset of $\mathbb{R}$. An open finite or infinite interval I is called a component interval of S if $I \subseteq S$ and if $\nexists$ an open interval J such that $I \subset J \subseteq S$.*

Theorem 1.14. *Let I denote a component interval of the open set S. If $x \in S$, then $\exists I \ni x \in I$. If $x \in I$, then $x \notin J$ where J is any other component interval of S.*

Proof. Since S is open, for any $x \in S$ there exists I an open interval such that $x \in I$ and $I \subseteq S$. There may be many such intervals, but the largest is $I_x = (a(x), b(x))$, where $a(x) = \inf\{a : (a, x) \subseteq S\}$, $b(x) = \sup\{b : (x, b) \subseteq S\}$. Note, a may be $-\infty$ and b may be $+\infty$. There is no open interval $J \ni I_x \subset J \subseteq S$ and by definition I_x is a component interval of S. If J_x is another component interval containing x, $I_x \cup J_x$ is an open interval $\ni I_x \cup J_x \subseteq S$. By definition of component inteval $I_x \cup J_x = I_x$ and $I_x \cup J_x = J_x$, so $I_x = J_x$. ■

Theorem 1.15. *Let $S \subseteq \mathbb{R}$ be open with $S \neq \emptyset$. Then $S = \cup_{n=1}^{\infty} I_n$ where $\{I_1, I_2, \dots\}$ is a collection of disjoint component intervals of S.*

Proof. If $x \in S$, then x belongs to one, and only one, component interval I_x. Note that $\bigcup\limits_{x \in S} I_x = S$ and by the definition of component intervals and the proof of the previous theorem, the collection of component intervals is disjoint. (If x belongs to I_x and J_x, both component intervals, $I_x = J_x$). Let $\{q_1, q_2, \dots\}$ be the collection of rational numbers (countable). In each component interval, there may be infinitely many of these, but among these there is exactly one with smallest index n. Define a function F, $F(I_x) = n$ if I_x contains the rational number x_n. If $F(I_x) = F(I_y) = n$ then I_x and I_y contain x_n, and $I_x = I_y$. Thus, the collection of component intervals is countable. ■

Remark 1.6. *Several collections of subsets of $\mathbb{R}$ generate $\mathcal{B}(\mathbb{R})$. In particular, we have:*

1. *Let $\mathcal{A}_1 = \{I : I = (a,b)$ is an interval (finite or infinite) with $-\infty \leq a < b \leq \infty\}$. Since (a,b) is open in $\mathbb{R}$, $\mathcal{A}_1 \subseteq \mathcal{O}_{\mathbb{R}}$ and $\sigma(\mathcal{A}_1) \subseteq \sigma(\mathcal{O}_{\mathbb{R}}) := \mathcal{B}_{\mathbb{R}}$. Every nonempty open set $O \subseteq \mathbb{R}$ can be written as $O = \cup_{\alpha=1}^{\infty} I_\alpha$, where I_α is a component interval of O. $I_\alpha \in \mathcal{A}_1$ $\forall \alpha$ and $I_\alpha \in \sigma(\mathcal{A}_1)$, hence $O \in \sigma(\mathcal{A}_1)$. Thus, $\mathcal{O}_{\mathbb{R}} \subseteq \sigma(\mathcal{A}_1)$ and $\sigma(\mathcal{O}_{\mathbb{R}}) \subseteq \sigma(\mathcal{A}_1)$. Together with $\sigma(\mathcal{A}_1) \subseteq \sigma(\mathcal{O}_{\mathbb{R}})$ gives $\sigma(\mathcal{O}_{\mathbb{R}}) = \sigma(\mathcal{A}_1)$.*

2. *Since $[a,b] = \cap_{n=1}^{\infty}(a - 1/n, b + 1/n)$, $[a,b] \in \sigma(\mathcal{A}_1)$. Hence the collection of closed intervals $\mathcal{A}_2 = \{I : I = [a,b], a, b \in \mathbb{R}\}$ is such that $\mathcal{A}_2 \subseteq \sigma(\mathcal{A}_1)$. Hence $\sigma(\mathcal{A}_2) \subseteq \sigma(\mathcal{A}_1)$. Also, since $(a,b) = \cup_{n=1}^{\infty}[a+1/n, b-1/n]$, $(a,b) \in \sigma(\mathcal{A}_2)$, hence the collection of open intervals $\mathcal{A}_1$ is such that $\mathcal{A}_1 \subseteq \sigma(\mathcal{A}_2)$ and $\sigma(\mathcal{A}_1) \subseteq \sigma(\mathcal{A}_2)$. Hence, $\sigma(\mathcal{A}_1) = \sigma(\mathcal{A}_2)$. But since, $\sigma(\mathcal{A}_1) = \sigma(\mathcal{O}_{\mathbb{R}})$, $\sigma(\mathcal{A}_2) = \sigma(\mathcal{O}_{\mathbb{R}})$.*

3. *Let* $\mathcal{A}_3 = \{I : I = (a, b] : -\infty \leq a < b < \infty\}$. *Note that since* $(a, b) = \cup_{n=1}^{\infty}(a, b - \frac{1}{n}]$ *we have that* $(a, b) \in \sigma(\mathcal{A}_3)$. *Consequently,* $\mathcal{A}_1 \subseteq \sigma(\mathcal{A}_3)$ *and* $\sigma(\mathcal{A}_1) \subseteq \sigma(\mathcal{A}_3)$. *Also, since* $(a, b] = \cup_{i=1}^{\infty}(a, b + \frac{1}{n})$ *we have that* $(a, b] \in \sigma(\mathcal{A}_1)$. *Consequently,* $\mathcal{A}_3 \subseteq \sigma(\mathcal{A}_1)$ *and* $\sigma(\mathcal{A}_3) \subseteq \sigma(\mathcal{A}_1)$. *Thus,* $\sigma(\mathcal{A}_3) = \sigma(\mathcal{A}_1)$.

4. *Using similar arguments, if* $\mathcal{A}_4 = \{I : I = (-\infty, a] : a \in \mathbb{R}\}$ *we have* $\sigma(\mathcal{A}_4) = \sigma(\mathcal{O}_{\mathbb{R}})$. *First, note that* $(-\infty, a] = \cap_{n=1}^{\infty}(-\infty, a + \frac{1}{n}) \in \sigma(\mathcal{A}_1)$. *Hence,* $\mathcal{A}_4 \subseteq \sigma(\mathcal{A}_1)$ *and* $\sigma(\mathcal{A}_4) \subseteq \sigma(\mathcal{A}_1)$. *Second, note that for* $a < b$

$$\begin{aligned}(a, b) &= (-\infty, b) \cap (a, \infty) = (-\infty, b) \cap (-\infty, a]^c \\ &= \left(\cup_{n=1}^{\infty}(-\infty, b - \frac{1}{n}]\right) \cap (-\infty, a]^c \in \sigma(\mathcal{A}_4).\end{aligned}$$

Hence, $\mathcal{A}_1 \subseteq \sigma(\mathcal{A}_4)$ *and* $\sigma(\mathcal{A}_1) \subseteq \sigma(\mathcal{A}_4)$ *and, together with the reverse set containment, we have* $\sigma(\mathcal{A}_1) = \sigma(\mathcal{A}_4)$.

Remark 1.7. *The collection of open sets of* $\mathbb{R}^n$ *will be denoted by* $\mathcal{O}_{\mathbb{R}^n}$. *Of course, this collection has the following properties:*

i) $\mathbb{R}^n, \emptyset \in \mathcal{O}_{\mathbb{R}^n}$

ii) $A_i \in \mathcal{O}_{\mathbb{R}^n}$, $i \in I$ *(arbitrary)* $\implies \underset{i \in I}{\cup} A_i \in \mathcal{O}_{\mathbb{R}^n}$

iii) $A_1, A_2, \cdots, A_m \in \mathcal{O}_{\mathbb{R}^n} \implies \cap_{i=1}^{m} A_i \in \mathcal{O}_{\mathbb{R}^n}$ *for* $m \in \mathbb{N}$.

Countable or arbitrary intersections of open sets do not belong to $\mathcal{O}_{\mathbb{R}^n}$.

Theorem 1.16. *Let* $\mathcal{O}_{\mathbb{R}^n}, \mathcal{C}_{\mathbb{R}^n}, \mathcal{K}_{\mathbb{R}^n}$ *be the collections of open, closed, and compact subsets of* $\mathbb{R}^n$. *Then,* $\sigma(\mathcal{O}_{\mathbb{R}^n}) = \sigma(\mathcal{C}_{\mathbb{R}^n}) = \sigma(\mathcal{K}_{\mathbb{R}^n})$. *By definition this is the Borel* σ*-algebra associated with* $\mathbb{R}^n$, *denoted by* $\mathcal{B}(\mathbb{R}^n)$.

Proof. From classical analysis if $A \subset \mathbb{R}^n$, then A compact $\iff$ A closed and bounded (Apostol, 1974, p. 59). Thus, $\mathcal{K}_{\mathbb{R}^n} \subseteq \mathcal{C}_{\mathbb{R}^n}$. Hence, by Theorem 1.3, $\sigma(\mathcal{K}_{\mathbb{R}^n}) \subseteq \sigma(\mathcal{C}_{\mathbb{R}^n})$.

Now, if $C \in \mathcal{C}_{\mathbb{R}^n}$ and $\bar{B}(\theta, k)$ is a closed ball with radius k centered at $\theta = (0, \ldots, 0)' \in \mathbb{R}^n$, i.e. $\bar{B}(\theta, k) = \{x \in \mathbb{R}^n : \|x\|_E \leq k\}$, then $C_k := C \cap \bar{B}(\theta, k)$ is closed and bounded. Closeness follows from the fact that complements of open sets are closed, and if A_i are open $(\cup A_i)^c = \cap A_i^c$ is closed and boundedness follows from $k \in \mathbb{N}$. Hence, $C_k \in \mathcal{K}_{\mathbb{R}^n}$ for all $k \in \mathbb{N}$. By construction, $C = \underset{k \in \mathbb{N}}{\cup} C_k$, thus $C \in \sigma(\mathcal{K}_{\mathbb{R}^n})$ and $\sigma(\mathcal{C}_{\mathbb{R}^n}) \subseteq \sigma(\mathcal{K}_{\mathbb{R}^n})$, so $\sigma(\mathcal{C}_{\mathbb{R}^n}) = \sigma(\mathcal{K}_{\mathbb{R}^n})$.

Since $\mathcal{C}_{\mathbb{R}^n} = (\mathcal{O}_{\mathbb{R}^n})^c$, we have that $\mathcal{C}_{\mathbb{R}^n} \subseteq \sigma(\mathcal{O}_{\mathbb{R}^n})$ and consequently $\sigma(\mathcal{C}_{\mathbb{R}^n}) \subseteq \sigma(\mathcal{O}_{\mathbb{R}^n})$. The converse $\sigma(\mathcal{O}_{\mathbb{R}^n}) \subseteq \sigma(\mathcal{C}_{\mathbb{R}^n})$ follows similarly to give $\sigma(\mathcal{C}_{\mathbb{R}^n}) = \sigma(\mathcal{O}_{\mathbb{R}^n})$. ■

The pairs $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ for $n \in \mathbb{N}$ are measurable spaces.

1.4 Measurable functions and random elements

Definition 1.9. *Let $(\Omega, \mathcal{F})$ and $(\mathbb{E}, \mathcal{E})$ be two measurable spaces. A function $f : (\Omega, \mathcal{F}) \to (\mathbb{E}, \mathcal{E})$ is said to be measurable if for all $A \in \mathcal{E}$, $f^{-1}(A) \in \mathcal{F}$.*

Remark 1.8. *If $(\Omega, \mathcal{F}, P)$ is a probability space we say that f is a random element. If, in addition, $(\mathbb{E}, \mathcal{E}) := (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ we will refer to $f : (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ as a random variable. We will normally represent random elements or random variables by uppercase roman letters, e.g., X or Y.*

The next theorem shows that measurability of a function f can be established by examining inverse images of sets in a collection that generates the measurable sets associated with the co-domain of f.

Theorem 1.17. *Let $\mathcal{C}$ be a collection of subsets of $\mathbb{E}$ such that $\sigma(\mathcal{C}) = \mathcal{E}$. Then, f measurable $\iff f^{-1}(\mathcal{C}) \subseteq \mathcal{F}$.*

Proof. $(\Rightarrow)$ First, assume f is measurable. f measurable $\iff$ for all $A \in \mathcal{E}$, $f^{-1}(A) \in \mathcal{F}$. In particular, let A be an element of $\mathcal{C}$, then $f^{-1}(A) \in \mathcal{F}$, hence $f^{-1}(\mathcal{C}) \subseteq \mathcal{F}$.

($\Leftarrow$) Second, assume that $f^{-1}(\mathcal{C}) \subseteq \mathcal{F}$, i.e., $f^{-1}(C) \in \mathcal{F}$, for all $C \in \mathcal{C}$. We must prove that $\forall A \in \mathcal{E}$, $f^{-1}(A) \in \mathcal{F}$ (or $f^{-1}(\mathcal{E}) \subseteq \mathcal{F}$). Let $\mathcal{G} = \{A \in \mathcal{E} : f^{-1}(A) \in \mathcal{F}\}$ and by construction $\mathcal{C} \subseteq \mathcal{G}$. If $\mathcal{G}$ is a σ-algebra, then $\sigma(\mathcal{C}) = \mathcal{E} \subseteq \mathcal{G}$. Also, by construction $\mathcal{G} \subseteq \mathcal{E}$, hence $\mathcal{E} = \mathcal{G}$, which is what must be proven.

We will show that $\mathcal{G}$ is a σ-algebra. Consider a sequence $A_1, A_2, \dots \in \mathcal{E}$ such that $f^{-1}(A_i) \in \mathcal{F}$ (i.e., $A_1, A_2 \dots \in \mathcal{G}$). Then, since $\mathcal{E}$ is a σ-algebra, $\cup_{i=1}^{\infty} A_i \in \mathcal{E}$. And since $f^{-1}(\cup_{i=1}^{\infty} A_i) = \cup_{i=1}^{\infty} f^{-1}(A_i)$, which is the union of elements in $\mathcal{F}$, $f^{-1}(\cup_{i=1}^{\infty} A_i) \in \mathcal{F}$. Now, if $A \in \mathcal{E}$ is such that $f^{-1}(A) \in \mathcal{F}$ (i.e., $A \in \mathcal{G}$), then $A^c \in \mathcal{E}$ and $f^{-1}(A^c) = f^{-1}(E) - f^{-1}(A) = \Omega - f^{-1}(A)$ which is in $\mathcal{F}$. Hence $\mathcal{G}$ is a σ-algebra. ■

Remark 1.9. 1. *We can take $\mathcal{A}_4 = \{(-\infty, a] : a \in \mathbb{R}\})$ (this is the collection $\mathcal{A}_4$ in Remark 1.6.4) and state that*

$$X : (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \sigma(\mathcal{C}) = \mathcal{B}(\mathbb{R})))$$

is a random variable if, and only if, $X^{-1}(\mathcal{C}) \subseteq \mathcal{F}$. Equivalently we can state X is a random variable if, and only if, $X^{-1}((-\infty, a]) = \{\omega \in \Omega : X(\omega) \leq a\} \in \mathcal{F}\ \forall a \in \mathbb{R}$.

2. *Since $X^{-1}((-\infty, a]) \in \mathcal{F}\ \forall a \in \mathbb{R}$ we can write $P(X^{-1}((-\infty, a])) = P \circ X^{-1}((\infty, a]) := P_X((-\infty, a])$, where $\circ$ indicates composition.*

Theorem 1.18. *Let $(\mathbb{X}, \mathcal{F}, \mu)$ be a measure space, $(\mathbb{E}, \mathcal{E})$ be a measurable space and $f : \mathbb{X} \to \mathbb{E}$ be a measurable function. Then,*

$$m(E) := \mu(f^{-1}(E)) \text{ for all } E \in \mathcal{E}$$

is a measure on $(\mathbb{E}, \mathcal{E})$.

Proof. We verify the two defining properties of measures. First, note that if $E = \emptyset$, $m(\emptyset) = \mu(f^{-1}(\emptyset)) = \mu(\emptyset) = 0$ since μ is a measure. Second, if $\{E_n\}_{n \in \mathbb{N}}$ is a pairwise disjoint collection of sets in $\mathcal{E}$ then

$$m\left(\underset{n\in\mathbb{N}}{\cup} E_n\right) = \mu\left(f^{-1}\left(\underset{n\in\mathbb{N}}{\cup} E_n\right)\right) = \mu\left(\underset{n\in\mathbb{N}}{\cup} f^{-1}(E_n)\right) = \sum_{n\in\mathbb{N}} \mu(f^{-1}(E_n)) = \sum_{n\in\mathbb{N}} m(E_n),$$

where the next to last equality follows from the fact that μ is a measure and the last equality follows from the definition of m. ■

If, in Theorem 1.18 we take $(\mathbb{X}, \mathcal{F}, \mu) := (\Omega, \mathcal{F}, P)$, $(\mathbb{E}, \mathcal{E}) := (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, and $f := X : (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ then $P_X := P \circ X^{-1}$ is a measure on $\mathcal{B}$.

The following definition follows directly from Remark 1.9.1.

Definition 1.10. *(Measurability of a real-valued function) A function $f : (\mathbb{X}, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is said to be $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable, or simply measurable, if for all $a \in \mathbb{R}$, the set $S_a = \{x \in \mathbb{X} : f(x) \leq a\} \in \mathcal{F}$.*

Remark 1.10. *Since $S_a \in \mathcal{F}$ and $\mathcal{F}$ is a σ-algebra, $S_a^c \in \mathcal{F}$. Hence, f is measurable if $S_a^c = \{x \in X : f(x) > a\} \in \mathcal{F}$. Also, consider $S_{a-1/n}^c = \{x \in \mathbb{X} : f(x) > a - 1/n\}$ and let $S_a' = \cap_{n=1}^{\infty}\{x \in \mathbb{X} : f(x) > a - 1/n\} = \{x \in \mathbb{X} : f(x) \geq a\}$. Clearly, by the properties of σ-algebras $S_a' \in \mathcal{F}$. Hence, f is measurable if $\{x \in X : f(x) \geq a\} \in \mathcal{F}$. Since, $\{x \in \mathbb{X} : f(x) < a\} = \{x \in \mathbb{X} : f(x) \geq a\}^c$, measurability could also be defined in terms of $\{x \in \mathbb{X} : f(x) < a\}$.*

Example 1.3.

1. *Let $f : \mathbb{X} \to \mathbb{R}$, such that for all $x \in \mathbb{X}$, $f(x) = c$, $c \in \mathbb{R}$. Let $a \in \mathbb{R}$ and consider $S_a^c = \{x \in \mathbb{X} : f(x) > a\} = \{x \in \mathbb{X} : c > a\}$. If $a \geq c$, $S_a^c = \emptyset$, and if $c > a$, $S_a^c = \mathbb{X}$. Since σ-algebras always contain $\emptyset$ and $\mathbb{X}$, $f(x) = c$ is measurable.*

2. *Let $E \in \mathcal{F}$ ($\mathcal{F}$ a σ-algebra). Recall that the indicator function of E is*

$$I_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

If $a \geq 1$, $S_a^c = \emptyset$; if $0 \leq a < 1$, $S_a^c = E$; if $a < 0$ $S_a^c = \mathbb{X}$. Since $\mathbb{X}, \emptyset \in \mathcal{F}$ (always) and $E \in \mathcal{F}$ by construction, I_E is measurable.

3. *Let* $f : (\mathbb{X}, \sigma(\mathcal{O}_{\mathbb{X}})) \to (\mathbb{Y}, \sigma(\mathcal{O}_{\mathbb{Y}}))$ *be continuous on* $\mathbb{X}$. *From classical analysis* $f^{-1}(\mathcal{O}_{\mathbb{Y}}) \subseteq \mathcal{O}_{\mathbb{X}}$. *Hence,* $f^{-1}(\mathcal{O}_{\mathbb{Y}}) \subseteq \sigma(\mathcal{O}_{\mathbb{X}})$ *and by Theorem 1.17* f *is measurable.*

4. *Let* $\mathbb{X} = \mathbb{R}$ *and* $\mathcal{F} = \mathcal{B}(\mathbb{R})$. *If* f *is monotone increasing, i.e.,* $\forall\, x < x'$, $f(x) \leq f(x')$, f *is measurable. Note that in this case,* $S_a^c = \{x : x > y \text{ for some } y \in \mathbb{R}\} = (y, \infty)$ *or* $S_a^c = \{x : x \geq y\}[y, \infty)$, *which are Borel sets.*

Theorem 1.19. *Let* f *and* g *be measurable real valued functions and let* $c \in \mathbb{R}$. *Then,* $cf, f^2, f + g, fg, |f|$ *are measurable.*

Proof. If $c = 0$, $cf = 0$ is a constant and consequently, measurable. If $c > 0$, then $\{x \in \mathbb{X} : cf(x) > a\} = \{x \in \mathbb{X} : f(x) > a/c\} \in \mathcal{F}$. Similarly for $c < 0$. If $a < 0$, $\{x \in \mathbb{X} : (f(x))^2 > a\} = \mathbb{X}$ and $\mathbb{X} \in \mathcal{F}$. If $a \geq 0$, $\{x \in \mathbb{X} : f^2(x) > a\} = \{x \in \mathbb{X} : f(x) > a^{1/2} \text{ or } f(x) < -a^{1/2}\} = \{x \in \mathbb{X} : f(x) > a^{1/2}\} \cup \{x \in \mathbb{X} : f(x) < -a^{1/2}\}$. The first set in the union is in $\mathcal{F}$ by assumption (f is measurable) and the second is in $\mathcal{F}$ by the arguments in Remark 1.10.

Now, $g(x) + f(x) > a \implies f(x) > a - g(x)$ which implies that there exists a rational number r such that $f(x) > r > a - g(x)$. Hence, $\{x \in \mathbb{X} : g(x) + f(x) > a\} = \underset{r\in\mathbb{Q}}{\cup}\{x \in \mathbb{X} : f(x) > r\} \cap \{x \in \mathbb{X} : g(x) > a - r\}$. Since the rational numbers are countable $\underset{r\in\mathbb{Q}}{\cup}$ is countable. Since f and g are measurable, and unions of countable measurable sets are measurable $\{x \in \mathbb{X} : g(x)+f(x) > a\} \in \mathcal{F}$. Note that $-f = (-1)f$. Hence if f is measurable, $-f$ is also measurable and so is $f + (-g) = f - g$.

Now, $fg = 1/2[(f + g)^2 - (f^2 + g^2)]$. Since $f^2, g^2, f + g, f - g$ and cf are measurable, if f, g are measurable, so is fg.

Lastly, $\{x \in \mathbb{X} : |f(x)| > a\} = \{x \in \mathbb{X} : f(x) > a \text{ or } f(x) < -a\} = \{x \in \mathbb{X} : f(x) > a\} \cup \{x \in \mathbb{X} : f(x) < -a\} = \{x \in \mathbb{X} : f(x) > a\} \cup \{x \in \mathbb{X} : -f(x) > a\}$. Since f and $-f$ are measurable, $\{x \in \mathbb{X} : |f(x)| > a\} \in \mathcal{F}$. ■

Recall that if $\{x_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers

$$\liminf_{n\to\infty} x_n := \sup_{k\in\mathbb{N}} \inf_{j\geq k}\{x_j\} \text{ and } \limsup_{n\to\infty} x_n := \inf_{k\in\mathbb{N}} \sup_{j\geq k}\{x_j\}.$$

Theorem 1.20. *Let $f_i(x) : \mathbb{X} \to \mathbb{R}$ for $i = 1, 2, \ldots$ be measurable. Then $\sup\{f_1, \ldots, f_n\}$, $\inf\{f_1, \ldots, f_n\}$, $\sup_n f_n$, $\inf_n f_n$, $\limsup_n f_n$ and $\liminf_n f_n$ are all measurable functions.*

Proof. Let $h(x) = \sup\{f_1(x), \ldots, f_n(x)\}$. Then, $S_a = \{x \in \mathbb{X} : h(x) > a\} = \cup_{i=1}^n \{x : f_i(x) > a\}$. Consequently, since f_i is measurable, $S_a \in \mathcal{F}$. Similarly if $g(x) = \sup_{n\in\mathbb{N}} f_n(x)$, $S_a = \{x \in \mathbb{X} : g(x) > a\} = \cup_{n=1}^\infty \{x : f_n(x) > a\} \in \mathcal{F}$. The same argument can be made for inf. Since $\limsup_{n\to\infty} f_n = \inf_{n\geq 1} \sup_{k\geq n} f_k$, $\limsup f_n$ is measurable. The same for $\liminf_{n\to\infty} f_n$. ■

As in Remark 1.6 several collections of subsets of $\mathbb{R}^n$ can generate the Borel sets of $\mathbb{R}^n$. Of particular interest are $\mathcal{I}^n = \{R^n = \times_{i=1}^n (a_i, b_i] : a_i, b_i \in \mathbb{R}\}$, $\mathcal{I}^{n,o} = \{R^{n,o} = \times_{i=1}^n (a_i, b_i) : a_i, b_i \in \mathbb{R}\}$, $\mathcal{I}^n_{\mathbb{Q}} = \{R^n_{\mathbb{Q}} = \times_{i=1}^n (a_i, b_i] : a_i, b_i \in \mathbb{Q}\}$ and $\mathcal{I}^{n,o}_{\mathbb{Q}} = \{R^{n,o}_{\mathbb{Q}} = \times_{i=1}^n (a_i, b_i) : a_i, b_i \in \mathbb{Q}\}$. In all cases, when $b_i \leq a_i$ we take $(a_i, b_i) = (a_i, b_i] = \emptyset$ and whenever an interval in any of the the cartesian products is empty, the cartesian product is empty.

Theorem 1.21. $\mathcal{B}(\mathbb{R}^n) = \sigma(\mathcal{I}^n) = \sigma(\mathcal{I}^{n,o}) = \sigma(\mathcal{I}^n_{\mathbb{Q}}) = \sigma(\mathcal{I}^{n,o}_{\mathbb{Q}})$.

Proof. Note that every open rectangle $R^{n,o}$ is an open set. To see this, choose any $x \in R^{n,o}$. Since (a_i, b_i) is open for all i and $n \in \mathbb{N}$, there exists $\delta > 0$ such that $(x_i - \delta, x_i + \delta) \subset (a_i, b_i)$ for all i. Let $B(x, \delta) = \{y : \|y - x\| < \delta\}$. Then, $\|y - x\| < \delta \iff \sum_{i=1}^n (y_i - x_i)^2 < \delta^2 \implies (y_i - x_i)^2 < \delta^2 - \sum_{j\neq i}^n (y_j - x_j)^2 < \delta^2 \implies |y_i - x_i| < \delta \iff y_i \in (x_i - \delta, x_i + \delta) \subset (a_i, b_i)$ for all i. Hence, $B(x, \delta) \subset R^{n,o}$. Since, $\mathcal{I}^{n,o}_{\mathbb{Q}} \subseteq \mathcal{I}^{n,o} \subseteq \mathcal{O}_{\mathbb{R}^n}$, we have $\sigma(\mathcal{I}^{n,o}_{\mathbb{Q}}) \subseteq \sigma(\mathcal{I}^{n,o}) \subseteq \sigma(\mathcal{O}_{\mathbb{R}^n}) = \mathcal{B}(\mathbb{R}^n)$.

Now, let $U \in \mathcal{O}_{\mathbb{R}^n}$ and consider $\underset{R\in\mathcal{I}^{o,n}_{\mathbb{Q}},\, R\subseteq U}{\cup} R$. Clearly, by construction $U \subseteq \underset{R\in\mathcal{I}^{o,n}_{\mathbb{Q}},\, R\subseteq U}{\cup} R$. Now, choose $x \in U$. Since U is open, there exists $B(x, \epsilon) \subset U$. Choose $R^{n,o}$ containing x such that $R^{n,o} \subset B(x, \epsilon)$ and further shrink this rectangle, by choosing $R^{n,o}_{\mathbb{Q}}$ containing

x such that $R^{n,o}_{\mathbb{Q}} \subset R^{n,o} \subset B(x,\epsilon)$. Hence, $\underset{R \in \mathcal{I}^{n,o}_{\mathbb{Q}},\, R \subseteq U}{\cup} R \subseteq U$ and we conclude that $U = \underset{R \in \mathcal{I}^{n,o}_{\mathbb{Q}},\, R \subseteq U}{\cup} R$. Since there are at most countably many rectangles in the union defining U, we have $U \in \mathcal{O}_{\mathbb{R}^n} \subseteq \sigma(\mathcal{I}^{n,o}_{\mathbb{Q}})$ and $\sigma(\mathcal{O}_{\mathbb{R}^n}) \subseteq \sigma(\mathcal{I}^{n,o}_{\mathbb{Q}})$. Hence, $\sigma(\mathcal{O}_{\mathbb{R}^n}) = \sigma(\mathcal{I}^{n,o}_{\mathbb{Q}}) = \sigma(\mathcal{I}^{n,o})$.

Since $R^n_{\mathbb{Q}} = \cap_{i \in \mathbb{N}} (a_1 - 1/i, b_1) \times \cdots (a_n - 1/i, b_n)$, $R^{n,o}_{\mathbb{Q}} = \cup_{i \in \mathbb{N}} [c_1 + 1/i, d_1) \times \cdots [c_n + 1/i, d_n)$, $R^n = \cap_{i \in \mathbb{N}} (a_1 - 1/i, b_1) \times \cdots (a_n - 1/i, b_n)$ and $R^{n,o} = \cup_{i \in \mathbb{N}} [c_1 + 1/i, d_1) \times \cdots [c_n + 1/i, d_n)$ we have $\sigma(\mathcal{I}^{n,o}) = \sigma(\mathcal{I}^n)$ and $\sigma(\mathcal{I}^{n,o}_{\mathbb{Q}}) = \sigma(\mathcal{I}^n_{\mathbb{Q}})$,which completes the proof. ■