ECON 4818 - Econometrics
Professor Carlos Martins
Midterm 2
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INSTRUCTIONS: Your answers should be written on just one side of a sheet of paper. Use a new sheet of paper to start the answer of each question.

IMPORTANT: Please indicate your grading preference:

[ ] I want to be graded as indicated in the course syllabus, i.e., 30 points for midterm 1, 30 points for midterm 2, 40 points for the final exam.

[ ] I want to be graded using the alternative option, i.e., I will drop the lowest grade on the midterms, and my final exam will be worth 70 points.

1. Let $Y = AX + BZ$ where A and B are non-stochastic matrices of dimension $m \times n$ and $m \times p$, and X and Z are random vectors of dimension $n \times 1$ and $p \times 1$.

   (a) Obtain the the variance of Y, i.e., $V(Y)$ as a function of the variance of X, Z and the covariance of X and Z.

   (b) From you work on (a) , is the covariance of X and Z the same as the covariance of Z and X? Explain.

   Hint: The variance of an arbitrary vector W of dimension $K \times 1$ is defined to be $V(W) = E(W - E(W))(W - E(W))^T$. The covariance between an arbitrary vector W of dimension $k \times 1$ and another arbitrary vector L of dimension $\ell \times 1$ is defined to be $\text{Cov}(W, L) = E(W - E(W))(L - E(L))^T$.

   Answer: (a) by definition $V(Y) = E[(Y - E(Y))(Y - E(Y))^T]$. Substituting $Y = AX + BZ$ we obtain

$$
\begin{aligned}
V(Y) &= E[(AX + BZ - E(AX + BZ))(AX + BZ - E(AX + BZ))^T] \\
&= E[(AX + BZ - AE(X) - BE(Z))(AX + BZ - AE(X) - BE(Z))^T] \\
&= E[(A(X - E(X)) + B(Z - E(Z))(A(X - E(X)) + B(Z - E(Z)))^T] \\
&= AE[(X - E(X))(X - E(X))^T]A^T + AE[(X - E(X))(Z - E(Z))^T]B^T \\
&+ BE[(Z - E(Z))(X - E(X))^T]A^T + BE[(Z - E(Z))(Z - E(Z))^T]B^T \\
&= AV(X)A^T + A\text{Cov}(X, Z)B^T + B\text{Cov}(Z, X)A^T + BV(Z)B^T
\end{aligned}
$$

   (b) Clearly, $\text{Cov}(X, Z)$ and $\text{Cov}(Z, X)$ have different dimensions. In this respect, these are different matrices. However, examining the individual elements of these matrices reveals that $\text{Cov}(X, Z) = \text{Cov}(Z, X)^T$.

2. Question: Consider the following multivariate linear regression model:

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + U_i \text{ for } i = 1, 2, \cdots, n.$$

where $E(U_i|X_{i1}, X_{i2}) = 0$ and $V(U_i|X_{i1}, X_{i2}) = \sigma^2$ and $\{U_i\}_{i=1}^n$ is a sequence of independent and identically distributed random variables.

(a) Write this regression model in matrix format, i.e., $Y = X\beta + U$ and give the elements of Y, X, β and U. What are their dimensions?

(b) Write $S_n = \sum_{i=1}^n U_i^2$ as a function of Y, X and β. What is $\frac{\partial}{\partial \beta} S_n(\beta)$?

(c) Obtain $\hat{\beta}$ that satisfies $\frac{\partial}{\partial \beta} S_n(\hat{\beta}) = 0$. What assumption is needed on $X^T X$ to obtain this expression for $\hat{\beta}$? Give an example of a setting where this assumption would be violated.

(d) What are the dimensions of $X^T X$? Is this a symmetric matrix? Prove.

(e) Show that $E(\hat{\beta}) = \beta$ and obtain $V(\hat{\beta}|X)$.

(f) If the vector U is assumed to have a multivariate normal distribution with $U \sim \mathcal{N}\left(\vec{0}, \sigma^2 I_n\right)$, what can be said about the distribution of $\hat{\beta}$ conditional on X? Explain with mathematical arguments.