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NONPARAMETRIC ESTIMATION OF DISTRIBUTION FUNCTIONS: TWO–MEASUREMENT PROBLEM

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We propose a plug-in estimator for a distribution function of two independent, but heterogeneously distributed random variables, where one variable has a density and the other has only a distribution. No restrictive assumptions are imposed on the distribution function, and only mild smoothness conditions on the density are required. We show that the proposed estimator is asymptotically unbiased. Bibliography: 10 *titles.*

1 Introduction

Estimation of a distribution function associated with the sum of independent, but possibly heterogeneously distributed random variables has been an enduring topic of interest in probability and statistics. This results from the fact that sums of independent random variables emerge in various applied fields, including reliability theory, actuarial sciences, medicine and economics,

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and ancillary components of statistical models (see, for example, [1]–[6]).

In this paper, we propose an estimator for a distribution function of two independent random variables. We suppose that X and Z are independent random variables, where X has a distribution function F and Z has a density f. We consider the estimation of the distribution function H of $X+Z$ at $t \in \mathbb{R}$ based on two independent random samples $\{X_i\}_{i=1}^n$ and $\{Z_j\}_{j=1}^n$. From [7, Theorem 11.1] it follows that

$$H(t) := F_{X+Z}(t) = \int_{\mathbb{R}} F(t-x) f(x) dx. \tag{1.1}$$

Equation (1.1) suggests the plug-in estimator

$$\widehat{H}(t) := \int_{\mathbb{R}} \widehat{F}(t-x) \widehat{f}(x) dx,$$

where $\widehat{F}$ and $\widehat{f}$ are estimators for F and f based on $\{X_i\}_{i=1}^n$ and $\{Z_j\}_{j=1}^n$. The plug-in estimation of the integrands of a convolution was first proposed in [2] in the case where both X and Z have a common density (see also [4]). Here, we relax assumptions required in the extant literature as follows:

(a) F is not necessarily absolutely continuous and can have jump discontinuities,

(b) We require the only condition $f \in L_1(\mathbb{R})$ and a mild smoothness condition on f are required.

We construct an estimator for $\widehat{F}(x)$ that belongs to the class of estimators proposed in [9] in the case where $\widehat{f}(x)$ is the Rosenblatt–Parzen kernel density estimator. Here, x is not necessarily a point of continuity of f, but x is required to be a Lebesgue point of f (see [8, Theorem 5.6.2]). Using properties of $\widehat{f}(x)$ and $\widehat{F}(x)$, we obtain bounds on the absolute value of the bias of $\widehat{H}(t)$ and show that these bounds approach zero as $n \to \infty$. To obtain this result, a stronger local Hölder type condition is imposed on f.

An interesting particular case of our general setting, where X and Z have the same distribution, i.e., f is a density associated with F, and $\widehat{f}$, $\widehat{F}$ are estimators based on the single random sample $\{X_i\}_{i=1}^n$ (see [2] for motivation).

2 The Main Results

2.1. Estimator for $F(x)$ and its bias representation. Let $\{X_i\}_{i=1}^n$ be a random sample of observations on the random variable X with distribution function given by F. The following estimator of $F(x)$ was proposed in [9]:

$$\widehat{F}(x) = \frac{1}{n} \sum_{i=1}^{n} U\Big(\frac{X_i - x}{h}\Big),$$

where $x \in \mathbb{R}$, $h > 0$ is a bandwidth, U is known and satisfies the following assumptions.

Assumption 2.1. U is continuous on $\mathbb{R}$, $\lim\limits_{x\to-\infty} U(x) = 1$, $\lim\limits_{x\to\infty} U(x) = 0$, and $U(0) = 1$.

Here, we add the requirement $U(0) = 1$ to the assumptions made by [9], which allows us to eliminate the impact of a possible jump discontinuity at x given by $F(x) - \lim\limits_{\varepsilon \downarrow 0} F(x-\varepsilon) :=$

$F(x) - F(x-)$ for $\varepsilon > 0$ on the convergence rate of the estimator. We set

$$\varphi_U(N) := \max\{\sup_{z<-N} |U(z) - 1|, \sup_{z>N} |U(z)|\}, \quad N > 0,$$

$$\omega_-(x, \varepsilon) := \omega_-(F, x, \varepsilon) = F(x-) - F(x - \varepsilon),$$

$$\omega_+(x, \varepsilon) := \omega_+(F, x, \varepsilon) = F(x + \varepsilon) - F(x), \ \varepsilon > 0.$$

Since F is a distribution function, it is everywhere right continuous, and we have

$$\lim_{N\to\infty} \varphi_U(N) = 0, \quad \lim_{\varepsilon\to 0} \omega_-(x, \varepsilon) = \lim_{\varepsilon\to 0} \omega_+(x, \varepsilon) = 0, \quad x \in \mathbb{R}. \tag{2.1}$$

We note two links between the continuity moduli of F and H. First,

$$\omega_+(H, s, \varepsilon) = H(s + \varepsilon) - H(s) = \int_{\mathbb{R}} [F(s - x + \varepsilon) - F(s - x)]f(x)dx$$

$$= \int_{\mathbb{R}} \omega_+(F, s - x, \varepsilon)f(x)dx,$$

which shows that the right continuity modulus of H behaves better than that of F. Second, taking $t \in (y - \varepsilon, y)$ and letting $t \uparrow y$, from the dominated convergence theorem we have

$$H(t) - H(y - \varepsilon) = \int_{\mathbb{R}} [F(t - x) - F(y - x - \varepsilon)]f(x)dx$$

$$\to \int_{\mathbb{R}} [F((y - x)-) - F(y - x - \varepsilon)]f(x)dx = \int_{\mathbb{R}} \omega_-(F, y - x, \varepsilon)f(x)dx.$$

Thus,

$$\omega_-(H, y, \varepsilon) = \int_{\mathbb{R}} \omega_-(F, y - x, \varepsilon)f(x)dx.$$

For the Riemann–Stieltjes integral we use the notation

$$\int_{(a,b]} g(t)dF(t) \quad \text{or} \quad \int_{[a,b)} g(t)dF(t)$$

instead of

$$\int_a^b g(t)dF(t)$$

to account for possible jumps of F at points a or b.

Theorem 2.1. *Suppose that U satisfies Assumption 2.1. Then*

(a) *For all $x \in \mathbb{R}$ and $\delta \in (0, 1)$*

$$|E\widehat{F}(x) - F(x)| = \left| \int_{\mathbb{R}} U\Big(\frac{y - x}{h}\Big)dF(y) - F(x) \right| \leqslant \varphi_U(h^{\delta-1})$$

$$+ \|U\|_C[\omega_-(x, h^\delta) + \omega_+(x, h^\delta)] + \omega_-(x, h^\delta), \quad h > 0. \tag{2.2}$$

(b) *If, in addition, U satisfies*

$$U(z) := \begin{cases} 1 & z < -u, \\ 0, & z > u \end{cases} \tag{2.3}$$

for some $u > 0$, then the term $\varphi_U(h^{\delta-1})$ in Equation (2.2) can be omitted for all $0 < h < u^{-1/(1-\delta)}$.

Proof. (a) Splitting integrals, we have

$$\int_{\mathbb{R}} U\Big(\frac{y-x}{h}\Big) dF(y) - F(x) = \Bigg\{ \int_{(-\infty,x-\varepsilon)} + \int_{[x-\varepsilon,x)} + \int_{[x,x]} + \int_{(x,x+\varepsilon]} + \int_{(x+\varepsilon,\infty)} \Bigg\} U\Big(\frac{y-x}{h}\Big) dF(y)$$

$$- \Bigg\{ \int_{(-\infty,x-\varepsilon)} + \int_{[x-\varepsilon,x)} + \int_{[x,x]} \Bigg\} dF(y) = \int_{(-\infty,x-\varepsilon)} \Big[U\Big(\frac{y-x}{h}\Big) - 1 \Big] dF(y)$$

$$+ \int_{[x-\varepsilon,x)} \Big[U\Big(\frac{y-x}{h}\Big) - 1 \Big] dF(y) + (U(0)-1)(F(x) - F(x-)) + \int_{(x,x+\varepsilon]} U\Big(\frac{y-x}{h}\Big) dF(y)$$

$$+ \int_{(x+\varepsilon,\infty)} U\Big(\frac{y-x}{h}\Big) dF(y). \tag{2.4}$$

By Assumption 2.1, the terms in Equation (2.4) are bounded as follows:

$$\Bigg| \int_{(-\infty,x-\varepsilon)} \Big[U\Big(\frac{y-x}{h}\Big) - 1 \Big] dF(y) \Bigg| \leqslant \sup_{z<-\varepsilon/h} |U(z) - 1| F(x),$$

$$\Bigg| \int_{[x-\varepsilon,x)} \Big[U\Big(\frac{y-x}{h}\Big) - 1 \Big] dF(y) \Bigg| \leqslant (\|U\|_C + 1)\omega_-(x,\varepsilon),$$

$$(U(0)-1)(F(x) - F(x-)) = 0,$$

$$\Bigg| \int_{(x,x+\varepsilon]} U\Big(\frac{y-x}{h}\Big) dF(y) \Bigg| \leqslant \|U\|_C \omega_+(x,\varepsilon),$$

$$\Bigg| \int_{(x+\varepsilon,\infty)} U\Big(\frac{y-x}{h}\Big) dF(y) \Bigg| \leqslant \sup_{z>\varepsilon/h} |U(z)|(1 - F(x)).$$

These bounds and Equation (2.4) imply

$$\Bigg| \int_{\mathbb{R}} U\Big(\frac{y-x}{h}\Big) dF(y) - F(x) \Bigg| \leqslant \varphi_U\Big(\frac{\varepsilon}{h}\Big) + (\|U\|_C + 1)\omega_-(x,\varepsilon) + \|U\|_C \omega_+(x,\varepsilon).$$

Selecting $\varepsilon = h^\delta$, we complete the proof of (a). Assertion (b) immediately follows from (a) if $h^{\delta-1} > u$. □

Remark 2.1. Let $\delta \in (0,1)$. If $h \to 0$ as $n \to \infty$, the asymptotic unbiasedness of $\widehat{F}(x)$ follows from the fact that $\varphi_U(h^{\delta-1})$, $\omega_+(x,h^\delta), \omega_-(x,h^\delta) \to 0$. Unlike [9], x is not required to be a point of continuity of F.

2.2. Estimator for $f(x)$ and its bias representation. By [8, Theorem 5.6.2], if $f \in L_1^{loc}(\mathbb{R})$, then

$$f(x) = \lim_{r\downarrow 0} \frac{1}{2r} \int\limits_{(x-r,x+r)} f(y)dy$$

for almost every $x \in \mathbb{R}$. In this case, x is called a *Lebesgue point* of f. Setting

$$g_x(t) := \frac{1}{2}(f(x+t) + f(x-t)) - f(x),$$

it is easy to see that if x is a Lebesgue point of f, then

$$\lim_{r\downarrow 0} \frac{1}{r} \int\limits_0^r g_x(t)dt = 0$$

and if

$$\gamma_x(\varepsilon) := \sup_{0<r\leqslant\varepsilon} \frac{1}{r} \left| \int\limits_0^r g_x(t)dt \right|, \quad \lim_{\varepsilon\to 0} \gamma_x(\varepsilon) = 0 \quad \text{a.e.} \tag{2.5}$$

For a given random sample $\{Z_j\}_{j=1}^n$ associated with Z we estimate $f(x)$ by using the Rosenblatt kernel estimator:

$$\widehat{f}(x) = \frac{1}{nh} \sum_{j=1}^n K\Big(\frac{Z_j - x}{h}\Big),$$

where K is a known kernel and $h > 0$ is a bandwidth. We make the following assumption on K.

Assumption 2.2. The function $K \in L_1(\mathbb{R})$ is even and, on $[0,\infty)$, nonnegative, nonincreasing, right continuous, and has the locally Riemann integrable derivative K'. In addition,

$$\int\limits_{\mathbb{R}} K(t)dt = 1.$$

It is evident from Assumption 2.2 that

$$K(r)\frac{r}{2} \leqslant \int\limits_{r/2}^r K(t)dt \leqslant K\Big(\frac{r}{2}\Big)\frac{r}{2}$$

which implies

$$\lim_{r\to\infty} K(r) = \lim_{r\to\infty} rK(r) = 0. \tag{2.6}$$

Definition 2.1. (a) If $\operatorname{supp} K = [-a,a]$ for some $a > 0$, then for $\varepsilon > 0$ we set $h(\varepsilon) = \varepsilon/a$.

(b) If $\operatorname{supp} K$ is not compact, then from Equation (2.6) the set $\{h > 0 : K(\varepsilon/h)\varepsilon/h \leqslant 2\varepsilon^2\}$ is not empty and we set

$$h(\varepsilon) = \sup\left\{h > 0 : \frac{\varepsilon}{h} K\Big(\frac{\varepsilon}{h}\Big) \leqslant 2\varepsilon^2, \int\limits_{\varepsilon/h}^{\infty} K(t)dt \leqslant \varepsilon\right\}.$$

By the right continuity of K, we observe that, in Definition 2.1 (b),

$$\frac{1}{h(\varepsilon)}K\Big(\frac{\varepsilon}{h(\varepsilon)}\Big) \leqslant 2\varepsilon, \qquad \int\limits_{\varepsilon/h(\varepsilon)}^{\infty} K(t)dt \leqslant \varepsilon. \tag{2.7}$$

In addition, $h(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Precursors of the following theorem can be found in [10, Theorem 6] in the functional-theoretical setting and without estimates for the convergence rate.

Theorem 2.2. *Let K satisfy Assumption 2.2, and let x be a Lebesgue point of f. Then for $\varepsilon > 0$*

(a) *if* $\operatorname{supp} K$ *is not compact, then*

$$|E\widehat{f}(x) - f(x)| = \Big|\int\limits_{\mathbb{R}} K(z)f(x + h(\varepsilon)z)dz - f(x)\Big| \leqslant \gamma_x(\varepsilon) + \varepsilon(1 + f(x)),$$

(b) *if* $\operatorname{supp} K = [-a, a]$, *then* $|E\widehat{f}(x) - f(x)| \leqslant \gamma_x(\varepsilon)$.

Proof. (a) Let $h := h(\epsilon)$. Consider the difference

$$E\widehat{f}(x) - f(x) = \frac{1}{h}\int\limits_{\mathbb{R}} K\Big(\frac{y-x}{h}\Big)f(y)dy - \int\limits_{\mathbb{R}} f(x)K(y)dy = \frac{1}{h}\int\limits_{0}^{\infty} K\Big(\frac{r}{h}\Big)(f(x+r) - f(x))dr$$

$$+\frac{1}{h}\int\limits_{-\infty}^{0} K\Big(\frac{r}{h}\Big)(f(x+r) - f(x))dr = \frac{2}{h}\int\limits_{0}^{\infty} K\Big(\frac{r}{h}\Big)g_x(r)dr = \frac{2}{h}\Big(\int\limits_{0}^{\varepsilon} + \int\limits_{\varepsilon}^{\infty}\Big)K\Big(\frac{r}{h}\Big)g_x(r)dr.$$

Put

$$G(t) = \int\limits_{0}^{t} g_x(s)ds.$$

Integrating by parts, we find

$$\frac{1}{h}\int\limits_{0}^{\varepsilon} K\Big(\frac{r}{h}\Big)g_x(r)dr = \frac{1}{h}\int\limits_{0}^{\varepsilon} K\Big(\frac{r}{h}\Big)dG(r) = \frac{1}{h}K\Big(\frac{\varepsilon}{h}\Big)G(\varepsilon) - \int\limits_{0}^{\varepsilon} G(r)\frac{1}{h^2}K'(r)dr. \tag{2.8}$$

All the integrals in the above equation exist and integration by parts is valid because K' is Riemann integrable and G is continuous. Since $rK(r) \leqslant 1/2$, we have

$$\Big|\frac{\varepsilon}{h}K\Big(\frac{\varepsilon}{h}\Big)\frac{G(\varepsilon)}{\varepsilon}\Big| \leqslant \frac{1}{2}\gamma_x(\varepsilon). \tag{2.9}$$

For the second term on the right-hand of (2.8) we have

$$\Big|\frac{1}{h^2}\int\limits_{0}^{\varepsilon} G(r)K'\Big(\frac{r}{h}\Big)dr\Big| \leqslant \gamma_x(\varepsilon)\int\limits_{0}^{\varepsilon} \frac{r}{h^2}\Big|K'\Big(\frac{r}{h}\Big)\Big|dr = -\gamma_x(\varepsilon)\int\limits_{0}^{\varepsilon/h} sK'(s)ds$$

$$= \gamma_x(\varepsilon)\Big(-\frac{\varepsilon}{h}K\Big(\frac{\varepsilon}{h}\Big) + \int\limits_0^{\varepsilon/h} K(s)ds\Big) \leqslant \frac{1}{2}\gamma_x(\varepsilon). \tag{2.10}$$

Equations (2.8) and (2.10) imply

$$\Big|\frac{1}{h}\int\limits_0^{\varepsilon} K\Big(\frac{r}{h}\Big)g_x(r)dr\Big| \leqslant \gamma_x(\varepsilon). \tag{2.11}$$

Furthermore, using the monotonicity of K and Equation (2.7), we find

$$\Big|\frac{1}{h}\int\limits_{\varepsilon}^{\infty} K\Big(\frac{r}{h}\Big)g_x(r)dr\Big| \leqslant \int\limits_{\varepsilon}^{\infty}\Big|\frac{f(x+r)}{2}\Big|\frac{1}{h}K\Big(\frac{r}{h}\Big)dr + \int\limits_{\varepsilon}^{\infty}\Big|\frac{f(x-r)}{2}\Big|\frac{1}{h}K\Big(\frac{r}{h}\Big)dr$$

$$+ f(x)\int\limits_{\varepsilon}^{\infty}\frac{1}{h}K\Big(\frac{r}{h}\Big)dr \leqslant \frac{1}{2h}K\Big(\frac{\varepsilon}{h}\Big)\Big(\int\limits_{\varepsilon}^{\infty}|f(x+r)|dr + \int\limits_{\varepsilon}^{\infty}|f(x-r)|dr\Big)$$

$$+ f(x)\int\limits_{\varepsilon/h}^{\infty} K(s)ds \leqslant \varepsilon(\|f\|_{L_1} + f(x)) = \varepsilon(1+f(x)). \tag{2.12}$$

This bound and Equation (2.11) prove Assertion (a).

(b) If $\operatorname{supp} K = [-a, a]$, then, instead of (2.12), we have

$$\frac{1}{h}\int\limits_{\varepsilon}^{\infty} K\Big(\frac{r}{h}\Big)g_x(r)dr = \int\limits_{\varepsilon/h}^{\infty} K(s)g_x(sh)ds = 0$$

for $h(\varepsilon) = \varepsilon/a$, and the conclusion follows from Equation (2.11). $\square$

2.3. Estimator for $H(t)$. For given random samples $\{X_i\}_{i=1}^n$ and $\{Z_j\}_{j=1}^n$ on X and Z we propose the following plug-in estimator for $H(t)$:

$$\widehat{H}(t) = \int\limits_{\mathbb{R}} \widehat{F}(t-x)\widehat{f}(x)dx = \frac{1}{n^2h}\sum_{i,j=1}^{n}\int\limits_{\mathbb{R}} U\Big(\frac{X_i - t + x}{b}\Big)K\Big(\frac{Z_j - x}{h}\Big)dx,$$

where $b, h > 0$ are bandwidths and U and K satisfy Assumptions 2.1 and 2.2 respectively. We prove two theorems that give bounds for the absolute value of the bias of $\widehat{H}(t)$. The first theorem deals with the general case where two independent random samples on X and Z are available. The second one considers the case where f is a density associated with F and there is a common random sample $\{X_i\}_{i=1}^n$ that must be used in the construction of $\widehat{f}$ and $\widehat{F}$. The second case is more involved as the estimators $\widehat{f}$ and $\widehat{F}$ are not independent since a common sample is used in their construction. Thus, we start by establishing the following auxiliary lemma that is useful in the second case.

Lemma 2.1. *For $\varepsilon > 0$ we set*

$$\Gamma(\varepsilon) := \int\limits_{\mathbb{R}} \gamma_x(\varepsilon)dx.$$

If Assumptions 2.1 *and* 2.2 hold, U *satisfies* (2.3), *and* $\operatorname{supp} K = [-a, a]$ *for some* $a > 0$, *then*

$$\left|\frac{1}{h}\int\limits_{\mathbb{R}} E\Big(U\Big(\frac{X-t+x}{h}\Big)K\Big(\frac{X-x}{h}\Big)\Big)dx - F\Big(\frac{t}{2}\Big)\right| \leqslant \Gamma(\varepsilon) \tag{2.13}$$

for all

$$h \leqslant \begin{cases} \dfrac{t-2x}{2\max\{u,a\}}, & x < t/2, \\ \dfrac{2x-t}{u+a}, & x > t/2 \end{cases}, \tag{2.14}$$

as in Definition 2.1.

Proof. Let

$$I := \frac{1}{h}\int\limits_{\mathbb{R}} E\Big(U\Big(\frac{X-t+x}{h}\Big)K\Big(\frac{X-x}{h}\Big)\Big)dx.$$

If, making the change of variable, we get

$$\begin{aligned} I &= \frac{1}{h}\int\limits_{\mathbb{R}}\Bigg(\int\limits_{\mathbb{R}} U\Big(\frac{y-t+x}{h}\Big)K\Big(\frac{y-x}{h}\Big)f(y)dy\Bigg)dx \\ &= \int\limits_{\mathbb{R}}\Bigg(\int\limits_{\mathbb{R}} U\Big(z+\frac{2x-t}{h}\Big)K(z)f(x+hz)dy\Bigg)dx. \end{aligned}$$

Thus, we write

$$\begin{aligned} I - F\Big(\frac{t}{2}\Big) &= \int\limits_{x>\frac{t}{2}}\Bigg(\int\limits_{\mathbb{R}} U\Big(z+\frac{2x-t}{h}\Big)K(z)f(x+hz)dz\Bigg)dx \\ &\quad + \int\limits_{x<\frac{t}{2}}\Bigg(\int\limits_{\mathbb{R}} U\Big(z+\frac{2x-t}{h}\Big)K(z)f(x+hz)dz\Bigg)dx - \int\limits_{x<\frac{t}{2}} f(x)dx. \end{aligned} \tag{2.15}$$

For $x > t/2$ the inner integral equals

$$\int\limits_{-a}^{a} U\Big(z+\frac{2x-t}{h}\Big)K(z)f(x+hz)dz$$

and we can use the inequality $h \leqslant (2x-t)/u+a$ (see (2.14)). Then $z \geqslant -a$ implies $z + (2x-t)/h \geqslant -a+u+a = u$, so that the above integral vanishes and its integral over $(t/2, \infty)$ also vanishes.

Now, we show that the remaining terms in Equation (2.15) are bounded by $\Gamma(\varepsilon)$:

$$\int\limits_{x<\frac{t}{2}} \Bigg(\int\limits_{\mathbb{R}} U\Big(z + \frac{2x-t}{h}\Big) K(z) f(x+hz)dz \Bigg) dx - \int\limits_{x<\frac{t}{2}} f(x)dx$$

$$= \int\limits_{x<\frac{t}{2}} \Bigg(\int\limits_{\mathbb{R}} \Big[U\Big(z + \frac{2x-t}{h}\Big) - 1\Big] K(z) f(x+hz)dz \Bigg) dx$$

$$+ \int\limits_{x<\frac{t}{2}} \Bigg(\int\limits_{\mathbb{R}} K(z) f(x+hz)dz - f(x) \Bigg) dx. \tag{2.16}$$

We put $N = \max\{u, a\}$ and recall that $K(z) = 0$ for $z > N$, so that

$$\int\limits_{z>N} \Big[U\Big(z + \frac{2x-t}{h}\Big) - 1\Big] K(z) f(x+hz)dz = 0. \tag{2.17}$$

For $x < t/2$ the range of h is limited by Equation (2.14): $h \leqslant (t-2x)/2N$. Then $z < N$ implies $z + (2x-t)/h \leqslant N - 2N = -N \leqslant -u$, so

$$\int\limits_{z<N} \Big[U\Big(z + \frac{2x-t}{h}\Big) - 1\Big] K(z) f(x+hz)dz = 0. \tag{2.18}$$

Equations (2.17) and (2.18) show that the first integral on the right-hand of (2.16) vanishes. For the remaining part of Equation (2.16), Theorem 2.2 (b) gives

$$\Bigg| \int\limits_{x<\frac{t}{2}} \Bigg(\int\limits_{\mathbb{R}} K(z) f(x+hz)dz - f(x) \Bigg) dx \Bigg| \leqslant \int\limits_{\mathbb{R}} \gamma_x(\varepsilon)dx = \Gamma(\varepsilon).$$

The lemma is proved. □

We provide a bound for the absolute value of the bias of $\widehat{H}(t)$ in the case where two independent random samples are available, whereas X and Z have different distributions.

Theorem 2.3. *Let $\{X_i\}_{i=1}^n$ and $\{Z_j\}_{j=1}^n$ be independent random samples. If $\operatorname{supp} K = [-a, a]$ for $a > 0$, U satisfies Equation (2.3), and Assumptions 2.1 and 2.2 hold, then*

$$|E\widehat{H}(t) - H(t)| \leqslant (1 + \|U\|_C)\Gamma(\varepsilon) + \int\limits_{\mathbb{R}} \Omega(t-x, b) f(x)dx, \tag{2.19}$$

where $\Omega(x,h) = \|U\|_C[\omega_-(x, h^\delta) + \omega_+(x, h^\delta)] + \omega_-(x, h^\delta)$.

Proof. We write

$$\widehat{H}(t) = \int\limits_{\mathbb{R}} \widehat{F}(t-x)\widehat{f}(x)dx = \int\limits_{\mathbb{R}} \frac{1}{n}\sum_{i=1}^{n} U\Big(\frac{X_i - t + x}{b}\Big) \frac{1}{nh}\sum_{j=1}^{n} K\Big(\frac{Z_j - x}{h}\Big)dx$$

$$= \frac{1}{n^2 h} \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{\mathbb{R}} U\Big(\frac{X_i - t + x}{b}\Big) K\Big(\frac{Z_j - x}{h}\Big) dx.$$

Taking expectations and using the fact that $\{X_i\}_{i=1}^n$ and $\{Z_j\}_{j=1}^n$ are independent random samples, we get

$$E(\widehat{H}(t)) = E\bigg(\int_{\mathbb{R}} U\Big(\frac{X - t + x}{b}\Big) \frac{1}{h} K\Big(\frac{Z - x}{h}\Big) dx \bigg)$$

$$= \int_{\mathbb{R}} E\Big(U\Big(\frac{X - t + x}{b}\Big)\Big) \frac{1}{h} E\Big(K\Big(\frac{Z - x}{h}\Big)\Big) dx.$$

Then

$$|E(\widehat{H}(t)) - H(t)| = \bigg| \int_{\mathbb{R}} E\Big(U\Big(\frac{X - t + x}{b}\Big)\Big) \frac{1}{h} E\Big(K\Big(\frac{Z - x}{h}\Big)\Big) dx - \int_{\mathbb{R}} F(t - x) f(x) dx \bigg|$$

$$= \bigg| \int_{\mathbb{R}} \Big[E\Big(U\Big(\frac{X - t + x}{b}\Big)\Big) \frac{1}{h} E\Big(K\Big(\frac{Z - x}{h}\Big)\Big) - F(t - x) f(x) \Big] dx \bigg|$$

$$= \bigg| \int_{\mathbb{R}} \Big[E\Big(U\Big(\frac{X - t + x}{b}\Big)\Big) \Big(\frac{1}{h} E\Big(K\Big(\frac{Z - x}{h}\Big)\Big) - f(x) \Big)$$

$$+ \Big(E\Big(U\Big(\frac{X - t + x}{b}\Big)\Big) - F(t - x) \Big) f(x) \Big] dx \bigg|$$

$$\leqslant \int_{\mathbb{R}} \Big| E\Big(U\Big(\frac{X - t + x}{b}\Big)\Big) \Big| \Big| \frac{1}{h} E\Big(K\Big(\frac{Z - x}{h}\Big)\Big) - f(x) \Big| dx$$

$$+ \int_{\mathbb{R}} \Big| E\Big(U\Big(\frac{X - t + x}{b}\Big)\Big) - F(t - x) \Big| f(x) dx.$$

Since U satisfies Equation (2.3) and $\operatorname{supp} K = [-a, a]$ for $a > 0$, from Theorem 2.1 (b) and Theorem 2.2 (b) we have

$$|E(\widehat{H}(t)) - H(t)| \leqslant \int_{\mathbb{R}} \gamma_x(\varepsilon) \Big| E\Big(U\Big(\frac{X - t + x}{b}\Big)\Big) \Big| dx + \int_{\mathbb{R}} \Omega(t - x, b) f(x) dx.$$

Since $\Big| E\Big(U\Big(\frac{X-t+x}{b}\Big)\Big) \Big| \leqslant F(t - x) + \Omega(t - x, b)$ and noting that $\Omega(x, h) \leqslant \Big\| U \Big\|_C + 1$, we get

$$|E(\widehat{H}(t)) - H(t)| \leqslant \int_{\mathbb{R}} \gamma_x(\varepsilon)(F(t - x) + \Omega(t - x, b)) dx + \int_{\mathbb{R}} \Omega(t - x, b) f(x) dx$$

$$\leqslant (1 + \|U\|_C) \int_{\mathbb{R}} \gamma_x(\varepsilon) dx + \int_{\mathbb{R}} \Omega(t - x, b) f(x) dx = (1 + \|U\|_C) \Gamma(\varepsilon) + \int_{\mathbb{R}} \Omega(t - x, b) f(x) dx.$$

The theorem is proved. $\square$

Remark 2.2. Suppose that f satisfies a local Hölder type condition, i.e., $|f(x+t)-f(x)| \leqslant c(x)|t|^\alpha$ for all $|t| \leqslant \varepsilon_0$ and $x \in \mathbb{R}$, where $c(x)$ is an integrable nonnegative function on $\mathbb{R}$ and $\alpha \in (0,1)$. Then for $t \in (0, \varepsilon_0)$

$$\left|\frac{f(x+t)+f(x-t)}{2} - f(x)\right| \leqslant \left|\frac{f(x+t)-f(x)}{2}\right| + \left|\frac{f(x-t)-f(x)}{2}\right| \leqslant c(x)t^\alpha.$$

Then

$$\gamma_x(\varepsilon) = \sup_{0<h\leqslant\varepsilon} \frac{1}{h}\left| \int_0^h \Big(\frac{f(x+t)+f(x-t)}{2} - f(x)\Big)dt\right|$$

$$\leqslant c(x) \sup_{0<h\leqslant\varepsilon} \frac{1}{h}\int_0^h t^\alpha dt = c(x) \sup_{0<h\leqslant\varepsilon} \frac{1}{h}\frac{h^{\alpha+1}}{(\alpha+1)} = c(x)\frac{\varepsilon^\alpha}{\alpha+1}, \quad \varepsilon \leqslant \varepsilon_0.$$

Hence

$$\Gamma(\varepsilon) = \int_{\mathbb{R}} \gamma_x(\varepsilon)dx \leqslant \frac{\varepsilon^\alpha}{\alpha+1}\int_{\mathbb{R}} c(x)dx = O(\varepsilon^\alpha).$$

Consequently, under the additional assumption that f satisfies a local Hölder type condition, $\widehat{H}(t)$ is asymptotically unbiased.

We now provide a bound for the absolute value of the bias of $\widehat{H}(t)$ when f is a density associated with F and estimation is based on a single random sample $\{X_i\}_{i=1}^n$.

Theorem 2.4. *Let f be a density associated with F, and let $\widehat{f}$ and $\widehat{F}$ be constructed by using the same random sample $\{X_i\}_{i=1}^n$. If* $\operatorname{supp} K = [-a,a]$ *for $a>0$, U satisfies* (2.3), *and Assumptions* 2.1 *and* 2.2 *hold, then*

$$|E\widehat{H}(t) - H(t)| \leqslant \frac{1}{n}[F(t/2)+\Gamma(\varepsilon)] + (1+\|U\|_C)\Gamma(\varepsilon) + \int_{\mathbb{R}} \Omega(t-x, h(\varepsilon))f(x)dx \tag{2.20}$$

for sufficiently small $\varepsilon > 0$.

Proof. By Lemma 2.1, for sufficiently small h

$$\left|\frac{1}{nh}\int_{\mathbb{R}} E\Big[U\Big(\frac{X-t+x}{h}\Big)K\Big(\frac{X-x}{h}\Big)\Big]dx\right| \leqslant \frac{1}{n}\Big(F\Big(\frac{t}{2}\Big)+\Gamma(\varepsilon)\Big). \tag{2.21}$$

As in the proof of Theorem 2.3,

$$\left|\frac{1}{h}\int_{\mathbb{R}} EU\Big(\frac{X-t+x}{h}\Big)EK\Big(\frac{X-x}{h}\Big)dx - \int_{\mathbb{R}} F(t-x)f(x)dx\right|$$

$$\leqslant \left|\int_{\mathbb{R}} EU\Big(\frac{X-t+x}{h}\Big)\Big[\frac{1}{h}EK\Big(\frac{X-x}{h}\Big) - f(x)\Big]dx\right|$$

$$
\begin{aligned}
&+\left|\int\limits_{\mathbb{R}} \Big[EU\Big(\frac{X-t+x}{h}\Big) - F(t-x)\Big] f(x)dx\right| \\
&\leqslant \int\limits_{\mathbb{R}} [F(t-x) + \Omega(t-x,h)]\gamma_\varepsilon(x)dx + \int\limits_{\mathbb{R}} \Omega(t-x,h)f(x)dx \\
&\leqslant (\|U\|_C + 1)\Gamma(\varepsilon) + \int\limits_{\mathbb{R}} \Omega(t-x,h)f(x)dx.
\end{aligned}
$$

The theorem follows from Equation (2.21) and the last bound. □

As in the case of Theorem 2.3, the asymptotic unbiasedess of $\widehat{H}(t)$ in this case follows from the assumption that f satisfies a local Hölder type condition.

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