Boolean algebras over partially ordered sets

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Abstract

Being the crossroads between Algebra, Topology, Logic, Set Theory and the Theory of Order; the class of Boolean algebras over partially ordered sets were look at as one of the sources, providing over time, new insights in Boolean algebras. Some constructions and their interconnections will be discussed, motivating along the way a list of open problems.

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1 Introduction

A Boolean algebra is an algebraic structure (a collection of elements and operations on them obeying defining axioms) that captures essential properties of both set operations and logic operations.

Examples:

1. Power set algebras: $(\wp(X), \cup, \cap, -, \emptyset, X)$

2. From logic: The Lendenbaum-Tarski algebra

3. From analysis: The algebra of projections

4. From topology: The regular open algebra

Stone representation for Boolean algebras

Every BA, $(A, +, ., -, 0, 1)$, *is isomorphic to an algebra of sets: There is an algebra-embedding* s *from* A *into the power set algebra of* $Ultra(A)$ *given by:*

$$s(a) = \{U \in Ultra(A) : \ a \in U\}$$

where $Ultra(A)$ is the set of ultrafilters of A.

Countable Boolean algebras (Mostowski-Tarski (1939))

Any countable Boolean algebras is generated by a chain $(C, <)$ *(i.e., a linear ordering).*

2 Interval algebras

Let $(L,<)$ be a linear ordering. For $t \in L$, set $b_t := [t, \rightarrow)$. The subalgebra of $(\wp(L), \cup, \cap, -, \emptyset, L)$ generated by $\langle b_t : t \in L \rangle$ is call ***the interval algebra over*** L and shall be denoted by $\mathbf{Int}(L)$. For technical reasons, we set $L^{\circ} = L \cup \{\infty\}$ with $\infty \notin L$

1 Algebra

Each non zero element, x, of $Int(L)$ has a unique decomposition under each of the following form

$$x = \cup_{i=0}^{n}[t_i,\, s_i[\ \ for\ \ t_i < s_i \ and\ s_i \in L^{\circ} \quad (1)$$

$$\mu^{+}(x) := |supp^{+}(x)| := |\{t_i :\ i < n\}| \quad (2)$$

$$x = \Delta_{i=0}^{n} b_{t_i}\ \ for\ \ t_i \in L \quad (3)$$

2 Topology

The Stone space of $Int(L)$ is homomeorphic to the set of f initial segments $I(L)$, of $(L,<)$, endowed with the topology inherited from 2^L.

$$\phi :\ Ultra(Int(L)) \longrightarrow I(L)$$

$$\phi(U) = \{t :\ b_t \in U\}$$

3 (Posets/ lattices)

$(Int(L),<)$ is a distributive complemented lattice, where $<$ the canonical boolean ordering on $Int(L)$.

4 Combinatorics

1. Let the set of elements that are in $x(\neq 0)$ be its positive support, $\mu^{+}(x)$, in $Int(L)$, when it is written in its normal form. Then For any natural number $m \geq 3$ and $x_1, \ldots, x_m$ in $int(L)$ pairwise disjoint we have:

$$\mu^{+}(\sum_{i=1}^{m} x_i) = \sum_{1 \leq i < j \leq m} \mu^{+}(x_i + x_j) - (m-2)\sum_{i=1}^{m} \mu^{+}(x_i)$$

2. (M. Rubin) *If B is a subalgebra, of $Int(L)$, of size κ uncountable regular, then B has a chain or an anti chain of size κ.*

5 Structure/generators

If $Int(L)$ has a chain of size κ uncountable regular, then $(L, <)$ has a chain , well ordered or anti-well ordered, of size κ.

6 Special properties

*A Boolean algebra B is **superatomic** if every nontrivial homomorphic image of B has an atom ($a \in B$ is an atom if $0 < a$ and there is no $x \in B$ so that $0 < x < a$).*

The **Cantor-Bendixon** rank:

Let A be a Boolean algebra, denote by $I(A)$ the ideal of A generated by the atoms of A. For any ordinal α, we define a sequence I_α of A as follows:

$$A_\alpha = A/I_\alpha$$

the α^{th} derivative of A, and let

$$\pi_\alpha : A \longrightarrow A_\alpha$$

be canonical. Define $I_0 = \{0\}$, $I_{\alpha+1} = \pi_\alpha^{-1}[I(A_\alpha)]$, and for λ a limit ordinal, $I_\lambda = \bigcup_{\alpha<\lambda} I_\alpha$. Finally, put $I_\infty = \bigcup_\alpha I_\alpha$.

If A is not trivial, the first $\alpha(A)$ so that $A_{\alpha(A)}$ is finite, is called the Cantor-Bendixon rank of A.

(Bonnet-Rubin-Sikaddour) *Every superatomic (subalgebra of)interval algebra is generated by a well founded lattice.*

Say that A satisfies the **qf**-property whenever every quotient of A is a factor i.e., if $\pi : A \twoheadrightarrow B$ then there is Q so that $A \simeq B \times Q$. Set $X := Ultra(Int(L))$, the Stone space of $Int(L)$.

(Bekkali-Bonnet-Rubin) *An interval algebra $Int(L)$ satisfies the **qf**-property iff $X := Ultra(Int(L))$ is homeomorphic to $\alpha + 1 + \sum_{i<n} \kappa_i + 1 + \lambda_i^\star$, where α is any ordinal, $n < \omega$, for every $i < \omega$, κ_i, λ_i are uncountable regular cardinals so that $\kappa_{i \geq \lambda_i}$ and if $n > 0$ then $\alpha \geq Max(\{\kappa_i : i < n\}).\omega$*

7 Sub(Int(L))/Hereditary properties

For $A, B \in Sub(Int(L))$, define:

$$A \vee B := \textit{the algebra generated by } (A \cup B), \quad A \wedge B := A \cap B.$$

(S. Todorcevic, M. Rubin)
For any $B \subseteq Sub(Int(L))$, $Sub(B)$ *is a complemented lattice.*

(M. Rubin)
Every B *in* $Sub(Int(L))$ *is a retractive algebra i.e., if* $\pi : B \twoheadrightarrow Q$ *there is* $r : Q \longrightarrow B$ *so that* $\pi \circ r = id_Q$.

Recall that a pseudo-tree $(T, <)$ is a partially ordered set so that $\downarrow t := \{s \in T : s < t\}$ is a chain.

(J. Nikiel, S. Purisch, L. Heindorf)
Every subalgebra of an interval algebra is isomorphic to a pseudo-treealgebra $B(T)$, *where* $(T, <)$ *is a pseudo-tree and* $B(T) := \langle b_t := \{s \in T : s \geq t\}\rangle$

(Bekkali)
Every pseudo-tree algebra $B(T)$ *embeds in very canonical way into an interval algebra.*

8 Relation with other classes

The algebra *finite-cofinite* $FC(\aleph_1)$ is not isomorphic to an interval algebra.

9 Cardinal functions

See Cardinal Invariants on BAs by JD Monk for different types of cardinal functions on BAs and Cardinals functions in topology-ten years later by I. Juhász

Problem 1: *Let, e.g., consider an interval algebra* $Int(L)$. *Is there a finite set of cardinal invariants that characterizes* $Int(L)$ *up to isomorphism?*

10 Set Theory

Problem 2: *Is it consistent to assume the existence of an uncountable interval algebra that is hereditary interval algebra?*

2 Pseudo-treealgebras

Results in these sections are in collaboration with D.Zhani.

Recall that a pseudo-tree $(T, <)$ is a partially ordered set so that $\downarrow t := \{s \in T : s < t\}$ is a chain.

Let b_t denotes the cone generated by t i.e., $b_t = \{s \in T : t \leq s\}$. The subalgebra of the $\langle \wp(T), \cap, \cup, 0, 1,' \rangle$ generated by $\langle b_t : t \in T \rangle$ is called the *pseudo-tree algebra generated by* $(T, <)$ *and denoted by* $B(T)$. The case of $B(T)$, where T is a well founded tree, were studied by G. Brenner.

1. Each non zero element of a pseudo-treealgebra $B(T)$ has a unique decomposition using the set of generators of cones $\langle b_t : t \in T \rangle$. Hence, one can define the positive support, μ^+, of a non zero element x, as in the case of interval algebra:

Let the set of elements of T that are in $x(\neq 0)$ be its positive support, $\mu^+(x)$, for $x \in B(T)$, when it is written in its normal form. Then For any natural number $m \geq 3$ and $x_1, \ldots, x_m$ in $int(L)$ pairwise disjoint we have:

$$\mu^+(\sum_{i=1}^{m} x_i) = \sum_{1 \leq i < j \leq m} \mu^+(x_i + x_j) - (m-2) \sum_{i=1}^{m} \mu^+(x_i) \qquad (\dagger)$$

Problem 3: *Let $(Q, <)$ be a poset and $B(Q)$ be the subalgebra of $\wp(Q)$ generated by $\langle b_t : t \in Q \rangle$. Assume there is l defined on $B(Q)$ satisfying $(\dagger)$. Is there a pseudo tree $(T, <)$ so that $B(Q) \simeq B(T)$ and $l = \mu^+$?*

2. Let $(T, <)$ be a pseudo-tree and set $X = Ultra(B(T))$. Recall that the set, $I_i(T)$, of *initial chains in* $(T, <)$ is a closed subspace of 2^T. Assuming, wlog, that $(T, <)$ has a single root, we have:

$$Ultra(B(T)) \simeq I_i(T)$$

3. *We say that a pseudo tree $(T, <)$ is* ***scattered*** *whenever:*
- *η the order type of the chain of rational numbers does not embed in $(T, <)$;*
- *The finite binary tree ${}^{<\omega}2$ does not embed in $(T, <)$ as a subtree.*

A pseuddo-treealgebra $B(T)$ is superatomic iff $(T, <)$ is scattered

Pseudo-treeagebras is an example of a **hereditarily** class of boolean algebras. Looking closely one may refine this statement as follows:

Let $(T, <)$ be a pseudo-tree and set $X = Ultra(B(T))(\simeq I_i(T))$. Denote by $cl(Z)$ the topological closure of Z in X. We say that $b \in X$ is a **bad point** whenever there are three disjoint sets A, B, C such that:

• Either at least A is discrete and uncountable so that $cl(A) = A \cup \{x\}$;

• Or, $cl(A) \cap cl(B) \cap cl(C) = \{x\}$, where $A = \{a_\alpha : \alpha < \kappa\}, B = \{b_\beta : \beta < \lambda\}$ are two linear orderings of uncountable cofinalities/coinalities converging to x and C is an infinite sequence so that $cl(C) = C \cup \{x\}$. Denote by $\mathbf{Bad}(X)$ the set of bad points in X.

If $B(T)$ is superatomic, then $B(T)$ is isomorphic to an interval algebra iff $\boldsymbol{Bad}(X) = \emptyset$

Problem 4: Is the previous statement true in more general setting?

Problem 5: (*Faithfulness of pseudo-tree algebras*)

Assume that there is a chain of type θ in the pseudo-treealgebra $\mathbf{B}(T)$. Is there a pseudo-tree T°, so that $\mathbf{B}(T) \cong \mathbf{B}(T^\circ)$, and T° embeds either a chain of type θ or $\theta^\star$.

Problem 6: ($n-$*hard pseudo-treealgebras*)

Let B be a boolean algebra and denote by $r_{pt}(B)$ the least ordinal β so that $B^{(\beta)} \simeq B(T)$ for some pseudo-tree T; $B^{(\beta)}$ being the β^{th} Cantor derivative of B.

Next denote by PT the class of all pseudo-treealgebras and set $r_{pt}(B) := \delta(B, PT)$. For example, if B is a pseudo-treealgebra, we set $\delta(B, PT) = 0 = r_{pt}(B)$, if $B \notin PT$ and $B^{(1)} \simeq B(T)$ for some pseudo tree T; we set $\delta(B, PT) = 1$, and we say that B is 1-***hard*** *pseudo-treealgebra.*

Characterize, in the class of superatomic boolean algebras, n-***hard*** *pseudo-treealgebras* for $n < \omega$.

3 Free poset algebras

Let $\langle P, \leq \rangle$ be a poset. *A free poset algebra over P* is a BA A having P as a set of generators such that:
(1) $p \leq_P q$ implies $p \leq_A q$,
(2) For every BA B and every mapping $f : P \longrightarrow B$, if $p \leq_P q$ implies that $f(p) \leq f(q)$, then there is a homomorphism $g : A \longrightarrow B$ such that $g \restriction P = f$.

Note that for P a poset consisting of isolated elements, that is with $\leq$ the identity, a free poset algebra over P is just a free BA over P. In particular, if P is infinite, then this algebra is atomless.

Proposition 1. *Let P be a poset, and let C be a boolean algebra freely generated by P. Let I be the ideal of C generated by $\{x \cdot -y : x, y \in P, \text{ and } x \leq_P y\}$. Then C/I is a free poset algebra over P.*

Proposition 2. *For any BA A the following conditions are equivalent:*
(i) A is isomorphic to a free poset algebra.
(ii) A has a set G of generators such that $0; 1 \in G$, and if $F; K$ are finite subsets of G and $\prod_{x \in F} x \cdot \prod_{x \in K} -y = 0$, then there exist $x \in F$ and $y \in K$ such that $x \cdot -y = 0$

For any poset P, a final segment of P is a subset M of P such that if $p \in M$ and $p \leq q$, then also $q \in M$. For any poset P, $Fs(P)$ is the collection of all final segments of P. Note that $\emptyset$ is a final segment of P.

Proposition 3. *Suppose that P is a poset. Then Fs(P) is a closed subspace of P. Moreover, clop(Fs(P)) is a free poset algebra on P.*

Proposition 4. *Every free poset algebra is a semigroup algebra.*

Let P be a poset. An antichain in P is a collection of pairwise incomparable elements of P. $Ant(P)$ is the set of all finite antichains of P. Note that $\emptyset \in Ant(P)$, and $\{p\} \in Ant(P)$ for all $p \in P$. We define a relation $\preceq$ on $Ant(P)$ by:

$$\sigma \preceq \tau \longleftrightarrow \forall p \in \sigma \exists q \in \tau (q \leq p)$$

Theorem 5. *Let P be a poset. Then:*
(i) $\langle Ant(P), \preceq \rangle$ is an upper semi lattice.
(ii) For each $p \in P$ let $f(p) = \{p\}$. Then f is an order anti-isomorphism of P into $Ant(P)$
(iii) $B(\langle Ant(P), \preceq \rangle)$is isomorphic to a free poset algebra over P.

4 Semi-group algebras

The following theorem gives a concrete construction of semi-group algebras. For let $(M, \wedge)$ be an idempotent semi-lattice with 0 and 1, and let A be the free Boolean algebra with free generators x_p for $p \in M$, and let I be the ideal generated by the set $\{(x_p \cdot x_q) \triangle x_{p\wedge q} : p, q \in M\}$.

Recall that a set $H \subseteq B$ is a disjunctive set whenever:
i. $0 \notin H$;
ii. For all $h, h_1, \dots, h_p \in H$ $[h \leq h_1 + \cdots + h_p \longrightarrow$ There is i so that $h \leq h_i]$

Recall that $(S, \cdot)$ is called a semi-lattice whenever " $\cdot$ " is commutative, associative, and $x^2 = x$ for all $x \in S$.

Theorem *The following statements are equivalent for any Boolean algebra B.*
i. *B is isomorphic to an upper semi-lattice algebra.*
ii. *B is generated by $H \subseteq B$ so that: $0 \notin H$, H is disjunctive, containing 1 and closed under multiplication.*

Theorem(D. Monk) *The following are equivalent.*

- ***i.*** *A/I is a semi-group algebra;*
- ***ii.*** *Every semi-group algebra is isomorphic to some A/I as above;*
- ***iii.*** *The Stone space $\mathcal{U}lt(A/I)$ is homomorphic to $\mathcal{F}(S)$, where $(S, \wedge)$ is a unitary meet semi-lattice.*

Corollary *Every semi-group algebra is isomorphic to an upper semi-lattice algebra.*

Note: If $(T, <)$ is a pseudo tree then $H := \{b_t : t \in T\}$ generates $B(T)$. Notice that $0 \notin H$, $1 \in H$, H is disjunctive ; but H is **not** closed under "$\cdot$". Hence one cannot draw the conclusion that $B(T)$ is a semigroup algebra and thus it is an upper semi-lattice by the previous theorem. Actually, there are examples where $B(T)$ is isomorphic to an upper semi-lattice algebra.

Problem 7: *Characterize atomless $B(T)$, T is a pseudo tree, that is isomorphic to an upper semi-lattice algebra.*

Problem $7^{'}$: ($n-$*almost pseudo-treealgebras*)

Denote by Pt the the class of pseudo-treealgebras. Call a tail algebra $B(T)(= \langle b_t : t \in T\rangle$ an ($\boldsymbol{n-}$***almost*** *pseudo-treealgebra*), whenever

$$b_u.b_v \;\; is\; a\;\; finite\; set\; of\;\; size\;\; n\;\; for\, all\;\; u,\; v\;\; incomparable\, in\, T.$$

• For n finite or countable, characterize upper semi lattices T so that $B(T)(= \langle b_t : t \in T \rangle$ is an ($\boldsymbol{n-almost}$ *pseudo-treealgebra*) isomorphic to a pseudo-treealgebra.

• Are ($\boldsymbol{n-almost}$ *pseudo-treealgebras*) retractive algebras? for $n \in \omega$.

5 Upper semi lattices algebras

Let $(T, <)$ be a partially ordered set and consider the subalgebra $B(T)$ of the power set of T, $\mathcal{P}(T)$ generated by $\{b_t : t \in T\}$, where $b_t := \{x \in T : t \leq x\}$.

Recall that an upper semi-lattice poset $(T, \leq)$ is so that $l.u.b.\{x, y\} := x \vee y$ exists in $(T, \leq)$ for all $x, y \in T$.

Next theorem characterizes $\mathcal{I}d(T)$, for any upper semi-lattice $(T, \leq)$.

Theorem *Let B be a Boolean algebra and set $X = \mathcal{U}lt(B)$ its Stone space. The following statements are equivalent.*

- **i.** *B is isomorphic to $B(T)$, where $(T, \leq)$ is an upper semi-lattice, with a least element.*
- **ii.** *X is homeomorphic to $\mathcal{I}d(T)$, the set of ideals of an upper semi-lattice T with a least element, endowed with Tychonoff's topology inherited from 2^T.*
- **iii.** *X is homeomorphic to $\mathcal{F}(S)$, the set of filters over S, where S is a unitary semi-lattice, endowed with Tychonoff's topology inherited from 2^S.*
- **iv.** *There is a multiplication " $\cdot$ " on X so that $(X, \cdot)$ is a unitary semi-lattice and " $\cdot$ " is a continuous mapping on $X \times X$ (i.e., $(x, y) \longrightarrow x \cdot y$ is continuous).*

Theorem *For any poset $(T, \leq)$ the Stone space $\mathcal{U}lt(B(T))$ of $B(T)$ is $\overline{\mathcal{I}d(T)}$ up to a homeomorphism.*

5.1 Set of initial sections of a poset

Let $(P, \leq)$ be a poset, with a least element. Denote by $\mathcal{I}(P)$ the set of non empty initial sections of P. $\mathcal{I}(P)$ is a Boolean space.

In this section, we state conditions under which $\mathcal{I}d(T)$, where $(T, \leq)$ is a poset, can be represented by $\mathcal{I}(P)$, up to a homeomorphism, for some poset P and conversely.

Notice that, when P is a chain, we have $\mathcal{I}(P) = \mathcal{I}d(P)$.

Definition

Let $(T, \leq)$ be an upper semi-lattice. An element $a \in T$ is prime whenever for all $c, d \in T$ $(a \leq c \vee d \rightarrow a \leq c$ or $a \leq d)$. Prim(T) shall denote the set of prime elements of T.
Also, we say that Prim(T) is $\vee$-generating set for T, whenever for every $t \in T$

there are $t_1, \dots, t_n \in Prim(T)$ so that $t = t_1 \vee \dots \vee t_n$.

Theorem *For any poset $(P, \leq)$, with a least element, there exists an upper semi-lattice T, with a least element, so that $\mathcal{I}(P) \cong_{homeo} \mathcal{I}d(T)$ where $Prim(T)$ is a $\vee$-generating set of T.*

Theorem *For any upper semi-lattice T, with a least element, so that Prim(T) is $\vee$-generating set of T, there is a poset P so that $\mathcal{I}d(T) \cong_{homeo} \mathcal{I}(P)$.*

Proof.
Let T be an upper semi-lattice, with a least element t_0, so that Prim(T) is $\vee$-generating set of T, and set $P := \{\downarrow t : t \in Prim(T)\}$; then $(P, \subseteq)$ is a poset with a least element $\downarrow \{t_0\} = \{t_0\}$. Define φ from $\mathcal{I}d(T)$ into $\mathcal{I}(P)$ by $\varphi(I) = (\downarrow I) \cap P$, where $\downarrow I :=_{def} \{J \in \mathcal{I}d(T) : J \subseteq I\}$. Again it easy to see that $\mathcal{I}d(T) \cong_{homeo} \mathcal{I}(P)$.

Let$(T, \leq)$ be a poset with a least element. Previous two theorems give indeed, necessary and sufficient conditions on $\mathcal{I}d(T)$ and $\mathcal{I}(P)$ to be homeomorphic spaces. We state these conditions in the following corollary.

Corollary *For any ($\vee$-f.g.)-poset $(T, \leq)$, with a least element, the following statements are equivalent.*

i. *There is a poset $(P, \leq)$, with a least element, so that $\mathcal{I}d(T) \cong_{homeo} \mathcal{I}(P)$.*

ii. *There is an upper semi-lattice T', with a least element, so that $\mathcal{I}d(T) \cong_{homeo} \mathcal{I}d(T')$, where Prim(T′) is a $\vee$-generating set of T'.*

7 Convex algebras

2^T shall be identified with the power set of T denoted by $\mathcal{P}(T)$. 2^T is a Boolean space endowed with Tychonoff's topology for which $\langle \mathcal{U}(A,B) : A, B \in [T]^{<\omega} \rangle$, where $\mathcal{U}(A,B) := \{F \in 2^T : A \subseteq F \text{ and } F \cap B = \emptyset\}$, is a basis for its topology. Let $(P,<)$ be a poset and set:

$$Convex(P) := \{X : \textit{ is a convex set } (P,<)\}$$

Recall that $X \subseteq P$ *is convex whenever* $(x, y \in X) \wedge (x < t < y \longrightarrow t \in X)$.

Claim *$Convex(P)$ is a closed subspace of $\{0,1\}^P$.*

Definition
Let $(P,<)$ be a poset. ***Clop***$(Convex(P))$ *is called the* ***convex algebra over*** P *denoted by $Convalg(P)$.*

9 Open Problems

For a partially ordered set (poset) $\langle P, < \rangle$, set $dim\langle P, < \rangle = \kappa$ to be the least λ such that $\langle P, < \rangle$ embeds in λ product of linear orderings $\langle P, < \rangle \preceq \prod_{i<\lambda} L_i$.

Problem 1.

Let T_n be a an upper semi lattice so that $dimT_n = n$. Is there an upper semi lattice T_{n-1} so that $dimT_{n-1} = n - 1$ and $B(T_n) \cong B(T_{n-1})$.

Problem 2.

How do sub-upper semi lattice algebras of an upper semi lattice algebras look like? E.g., characterize upper semi-lattices T such that $B(T) \leq B(\omega_1 \times \omega_1)$.

Problem 3.

It is consistent that any Tail algebra $B(P)$, P is a poset, embeds in (*canonical* way) an upper semi lattice algebra?

Problem 4.

Is it true that for any uncountable upper semi-lattice T, $B(T)$ has an uncountable chain or an uncountable anti-chain?

Problem 5.

It is consistent that for each $n \in \mathbb{N}$, there is an upper semi lattice $T^{\vee}$ of dimension n so that: $inc(B(T^{\vee})) = ln(B(T^{\vee})) = \aleph_1$.

Problem 6.

Let T be a poset so that each point, in $Id(T)$, has a countable local basis and $Id(T)$ has *no bad-points*. Is $B(T)$ isomorphic to an interval algebra?

Problem 7.

Let $B := Int(C)$ be an interval algebra over the chain C, and B_0, B_1 subalgebras of $B := Int(C)$. Call B_0, B_1 *relatively prime* in $B := Int(C)$ whenever:
$B \approx B_0 \times B_1$; $B_0 \wedge B_1 = \{0, 1\}$.
Is it true that if B_0 and B_1 are relatively prime in $Int(C)$ they are complements of each other in $Sub(Int(C))$ and vice versa?

Problem 8.

Characterize $Int(C)$ so that if D is a sub-chain of $Int(C)$ then there is a boolean algebra E such that: $Int(C) \simeq \langle D\rangle \times E$.

Problem 9.

Charecterize P so that $B(P) \simeq B(\mathrm{Id}(P))$.

Problem 10:

Is it consistent to assume the existence of an interval algebra $|B| = \aleph_1 = inc(B) = d(B)$ so that each of its subalgebra is isomorphic to an interval algebra.

Problem 11:

Is it consistent to assume the existence of an interval algebra B, $|B| = \kappa^+$, $inc(B) = \kappa$, $d(B) \neq \aleph_0, \kappa$; so that each of its subalgebra is isomorphic to an interval algebra.

Problem 12:

Is there an uncountable upper semi lattice algebra $B(T)$ which is a hereditarily upper semi-lattice algebra?

Problem 13:

Let $(T, <)$ be a upper semi lattice so that $dim(T) = p < \omega$. Is every subalgebra of $B(T)$ generated by an upper semi lattice T^* so that $dim(T^*) \leq p+1$?

Problem 14:

Characterize $(T, <)$ so that $B(T)$ is a subalgebra of $B(C_1 \times C_2 \times .... \times C_n)$ where $C_i's$ are chains.

Problem 15:

Do semi lattices $(T, <)$ of $dim(T) = 3$ are so that $B(T)$ is a subalgebra of an interval algebra?

Problem 16:

Characterize T, K so that $dim(T) = dim(K) + 1$ and $B(T) \simeq B(K)$.

Problem 17:

Denote by $Convalg(P)$, $B(T)$, $F(P)$, $Int(L)$, $B_{pt}(T)$ any convex boolean algebra, algebra generated by upper semi lattice T, free algebra over P, an interval algebra over L, and a pseutree-algebra over T respectively. Characterize:

Convalg $(P) \cong F(Q)$

Convalg $(P) \cong B(T))$

Convalg $(P) \cong B_{pt}(T)$

Convalg $(P) \cong Int(L)$

Problem 18:

Let $CPU(\mathbf{B})$ be the class of boolean algebras that are product of *three pairwise non isomorphic* elements from the classes respectively of convex algebras *C(B)*, pseudotree-algebras *P(B)*, upper semi-lattice algebras *U(B)*, respectively.

> *Is CPU($\mathbf{B}$) a new class different from the previously defined ones?*

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