Open problems in Boolean algebras over partially ordered sets

Mohamed Bekkali

University sidi mohamed ben abdullah (USMBA)
Fez, Morocco

bekka@menara.ma

Abstract

Being the crossroads between Algebra, Topology, Logic, Set Theory and the Theory of Order; the class of Boolean algebras over partially ordered sets were look at as one of the sources, providing over time, new insights in Boolean algebras. We state the following open problems, hoping that some of them will open new horizons in this direction towards understanding this class of BAS.

BLAST, Boulder, Co. June 2-6, 2010

Problem 1: *Let, e.g., consider an interval algebra $Int(L)$. Is there a finite set of cardinal invariants that characterizes $Int(L)$ up to isomorphism?*

Problem 2: *Is it consistent to assume the existence of an uncountable interval algebra that is hereditary interval algebra?*

Problem 3: *Let $(Q, <)$ be a poset and $B(Q)$ be the subalgebra of $\wp(Q)$ generated by $\langle b_t : t \in Q \rangle$. Assume there is l defined on $B(Q)$ satisfying ($\dagger$). Is there a pseudo tree $(T, <)$ so that $B(Q) \simeq B(T)$ and $l = \mu^+$?*

Pseudo-treeagebras is an example of a **hereditarily** class of boolean algebras. Looking closely one may refine this statement as follows:

Let $(T,<)$ be a pseudo-tree and set $X = Ultra(B(T))(\simeq I_i(T))$. Denote by $cl(Z)$ the topological closure of Z in X. We say that $b \in X$ is a **bad point** whenever there are three disjoint sets A, B, C such that:
• Either at least A is discrete and uncountable so that $cl(A) = A \cup \{x\}$;
• Or $cl(A) \cap cl(B) \cap cl(C) = \{x\}$, where $A = \{a_\alpha : \alpha < \kappa\}, B = \{b_\beta : \beta < \lambda\}$ are two linear orderings of uncountable cofinalities/coinalities converging to x and C is an infinite sequence so that $cl(C) = C \cup \{x\}$. Denote by $\mathbf{Bad}(X)$ the set of bad points in X.

If $B(T)$ is superatomic, then $B(T)$ is isomorphic to an interval algebra iff $\boldsymbol{Bad}(X) = \emptyset$

Problem 4: Is the previous statement true in more general setting?

Problem 5: (*Faithfulness of pseudo-tree algebras*)

Assume that there is a chain of type θ in the pseudo-treealgebra $\mathbf{B}(T)$. Is there a pseudo-tree T°, so that $\mathbf{B}(T) \cong \mathbf{B}(T^\circ)$, and T° embeds either a chain of type θ or $\theta^\star$.

Problem 6: ($n-$*hard pseudo-treealgebras*)

Let B be a boolean algebra and denote by $r_{pt}(B)$ the least ordinal β so that $B^{(\beta)} \simeq B(T)$ for some pseudo-tree T; $B^{(\beta)}$ being the β^{th} Cantor derivative of B. Next denote by PT the class of all pseudo-treealgebras and set $r_{pt}(B) := \delta(B, PT)$. For example, if B is a pseudo-treealgebra, we set $\delta(B, PT) = 0 = r_{pt}(B)$, if $B \notin PT$ and $B^{(1)} \simeq B(T)$ for some pseudo tree T; we set $\delta(B, PT) = 1$, and we say that B is 1-***hard*** *pseudo-treealgebra.*

Characterize, in the class of superatomic boolean algebras, n-***hard*** *pseudo-treealgebras* for $n < \omega$.

Problem 7: ($n-$*almost pseudo-treealgebras*)

Denote by Pt the the class of pseudo-treealgebras. Call a tail algebra $B(T)(= \langle b_t : t \in T\rangle$ an ($\boldsymbol{n-almost}$ *pseudo-treealgebra*), whenever

$$b_u.b_v \;\; is\ a\ \ finite\ set\ of\ size\ \ n\ \ for\ all\ \ u,\ v\ incomparable\, in\, T.$$

• For n finite or countable, characterize upper semi lattices T so that $B(T)(= \langle b_t : t \in T\rangle$ is an ($\boldsymbol{n-almost}$ *pseudo-treealgebra*) isomorphic to a pseudo-treealgebra.

• Are ($\boldsymbol{n-almost}$ *pseudo-treealgebras*) retractive algebras? for $n \in \omega$.

For a partially ordered set (poset) $\langle P, <\rangle$, set $dim\langle P, <\rangle = \kappa$ to be the least λ such that $\langle P, <\rangle$ embeds in λ product of linear orderings $\langle P, <\rangle \preceq \prod_{i<\lambda} L_i$.

Problem 8.

Let T_n be a an upper semi lattice so that $dim T_n = n$. Is there an upper semi lattice T_{n-1} so that $dim T_{n-1} = n-1$ and $B(T_n) \cong B(T_{n-1})$.

Problem 9.

How do sub-upper semi lattice algebras of an upper semi lattice algebras look like? E.g., characterize upper semi-lattices T such that $B(T) \leq B(\omega_1 \times \omega_1)$.

Problem 10.

It is consistent that any Tail algebra $B(P)$, P is a poset, embeds in (*canonical* way) an upper semi lattice algebra?

Problem 11.

Is it true that for any uncountable upper semi-lattice T, $B(T)$ has an uncountable chain or an uncountable anti-chain?

Problem 12.

It is consistent that for each $n \in \mathbb{N}$, there is an upper semi lattice $T^{\vee}$ of dimension n so that: $inc(B(T^{\vee})) = ln(B(T^{\vee})) = \aleph_1$.

Problem 13.

Let T be a poset so that each point, in $Id(T)$, has a countable local basis and $Id(T)$ has *no bad-points*. Is $B(T)$ isomorphic to an interval algebra?

Problem 14.

Let $B := Int(C)$ be an interval algebra over the chain C, and B_0, B_1 subalgebras of $B := Int(C)$. Call B_0, B_1 *relatively prime* in $B := Int(C)$ whenever:
$B \approx B_0 \times B_1$; $B_0 \wedge B_1 = \{0, 1\}$.
Is it true that if B_0 and B_1 are relatively prime in $Int(C)$ they are complements of each other in $Sub(Int(C))$ and vice versa?

Problem 15.

Characterize $Int(C)$ so that if D is a sub-chain of $Int(C)$ then there is a boolean algebra E such that: $Int(C) \simeq \langle D\rangle \times E$.

Problem 16.

Charecterize P so that $B(P) \simeq B(Id(P))$.

Problem 17:

Is it consistent to assume the existence of an interval algebra $|B| = \aleph_1 = inc(B) = d(B)$ so that each of its subalgebra is isomorphic to an interval algebra.

Problem 18:

Is it consistent to assume the existence of an interval algebra B, $|B| = \kappa^+$, $inc(B) = \kappa$, $d(B) \neq \aleph_0, \kappa$; so that each of its subalgebra is isomorphic to an interval algebra.

Problem 19:

Is there an uncountable upper semi lattice algebra $B(T)$ which is a hereditarily upper semi-lattice algebra?

Problem 20:

Let $(T, <)$ be a upper semi lattice so that $dim(T) = p < \omega$. Is every subalgebra of $B(T)$ generated by an upper semi lattice T^* so that $dim(T^*) \leq p + 1$?

Problem 21:

Characterize $(T, <)$ so that $B(T)$ is a subalgebra of $B(C_1 \times C_2 \times .... \times C_n)$ where $C_i's$ are chains.

Problem 22:

Do semi lattices $(T, <)$ of $dim(T) = 3$ are so that $B(T)$ is a subalgebra of an interval algebra?

Problem 23:

Characterize T, K so that $dim(T) = dim(K) + 1$ and $B(T) \simeq B(K)$.

Problem 24:

Denote by $Convalg(P)$, $B(T)$, $F(P)$, $Int(L)$, $B_{pt}(T)$ any convex boolean algebra, algebra generated by upper semi lattice T, free algebra over P, an interval algebra over L, and a pseutree-algebra over T respectively. Characterize:

Convalg $(P) \cong F(Q)$

Convalg $(P) \cong B(T))$

Convalg $(P) \cong B_{pt}(T)$

Convalg $(P) \cong Int(L)$

Problem 25:

Let $CPU(\mathbf{B})$ be the class of boolean algebras that are product of *three pairwise non isomorphic* elements from the classes respectively of convex algebras *C(B)*, pseudotree-algebras *P(B)*, upper semi-lattice algebras *U(B)*, respectively.

Is CPU($\mathbf{B}$*) a new class different from the previously defined ones?*

References

[1] M. Bekkali, Chains and antichains in interval algebras, J. Symbolic Logic 59 (1994), no. 3, 860-867.

[2]———, Pseudo treealgebras, Notre Dame J. Formal Logic 42 (2001), no. 2, 101-108.

[3] M. Bekkali, D. Zhani, Upper Semi-Lattice algebras and combinatorics; Set theory: Centre de recerca matemàtica Barcelona, 2003-2004 by Joan Bagaria, Stevo Todorcevic ; Trends in Mathematics, Birkhäuser.

[4] L. Heindorf, On subalgebras of interval algebras, Proceeding of the American Mathematical Society; Volume 125, Number 8, August 1997, Pages 2265-2274 S 0002-9939(97)03851-3

[5] S. Koppelberg, Minimally generated Boolean algebras, Order 5 (1989), no. 4, 393-406.

[6] S. Koppelberg and J. D. Monk, Pseudo-trees and Boolean algebras, Order 8 (1991/1992), no. 4, 359-374.

[7] J. D. Monk, Cardinal Functions on Boolean Algebras, Lectures in Mathematics ETH Z ürich, Birkhäuser Verlag, Basel, 1990.

[8] J. D. Monk and R. Bonnet (eds.), Handbook of Boolean Algebras. Vol. 3, North-Holland Publishing, Amsterdam, 1989. [7] A. Mostowski and A. Tarski, Boolesche ringe mit geordneter basis, Fund. Math. 32 (1939), 69-86 (German).

[9] J. Nikiel, Orderability properties of a zero-dimensional space which is a continuous image of an ordered compactum, Topology Appl. 31 (1989), no. 3, 269-276.

[10] J. Nikiel, S. Purisch, and L. B. Treybig, Separable zero-dimensional spaces which are continuous images of ordered compacta, Houston J. Math. 24 (1998), no. 1, 45-56.

[11] J. G. Rosenstein, Linear Orderings, Pure and Applied Mathematics, vol. 98, Academic Press, New York, 1982 .

[12] M. Rubin, R. Bonnet On poset Boolean algebras of scattered posets with finite width, Arch. math. logic 43, 467-476 (2004)