Canonical extensions of lattices

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Let L be a lattice and C a complete lattice with L isomorphic to a sublattice of C. Then C is a completion of L and

- C is a dense completion if any element of C can be expressed as both a meet of joins and join of meets of elements of L,
- C is a compact completion if for any filter F, and ideal I, of L if $\bigwedge F \leqslant \bigvee I$ then $F \cap I \neq \emptyset$.

If C is both a dense and compact completion of L it is called a canonical extension of L.

Result: Every lattice L has a canonical extension, and this is unique up to an isomorphism which fixes L.

(Gehrke and Harding, 2001)

History of canonical extensions:

1951 Jónsson & Tarski: canonical extensions for Boolean algebras with operators

1994 Gehrke & Jónsson: bounded distributive lattices with operators

2000 Gehrke & Jónsson: bounded distributive lattices with monotone operations

2001 Gehrke & Harding: bounded lattice expansions

2004 Gehrke & Jónsson: distributive lattices with arbitrary operations

2005 Dunn, Gehrke, Palmigiano: partially ordered sets

2009 Moshier & Jipsen: topological duality theorem for bounded lattices

Construction of the canonical extension

Using the filters, $\mathcal{F}(L)$, and ideals, $\mathcal{I}(L)$, of L, form $\mathcal{F}(L) \cup \mathcal{I}(L)$. This is the *intermediate structure*, ordered by:

- $F_1 \leqslant^* F_2 \iff F_2 \subseteq F_1$
- $I_1 \leqslant^* I_2 \iff I_1 \subseteq I_2$
- $F \leqslant^* I \iff F \cap I \neq \emptyset$
- $I \leqslant^* F \iff x \in I, y \in F \Longrightarrow x \leqslant y$

Then take the MacNeille completion of the intermediate structure.

L^δ is the canonical extension. $L^\delta \subseteq \wp\Big(\mathcal{F}(L) \cup \mathcal{I}(L)\Big)$.

Filter and ideal elements of L^δ

$p = \bigwedge F$, where $F \in \mathcal{F}(L)$, is a *filter* element

$u = \bigvee I$, where $I \in \mathcal{I}(L)$, is an *ideal* element

$F(L^\delta)$: filter elements of L^δ

$I(L^\delta)$: ideal elements of L^δ

$F(L^\delta)$ is order isomorphic to $\big(\mathcal{F}(L), \supseteq\big)$, and $I(L^\delta)$ is order isomorphic to $\big(\mathcal{I}(L), \subseteq\big)$.

$$\alpha : \mathcal{F}(L) \longrightarrow C,\ F \longmapsto \bigwedge e[F]$$

$$\beta : \mathcal{I}(L) \longrightarrow C,\ I \longmapsto \bigvee e[I]$$

This gives $\Big(\mathcal{F}(L) \cup \mathcal{I}(L), \leqslant^*\Big)$ order isomorphic to $\Big(F(L^\delta) \cup I(L^\delta), \leqslant\Big)$.

The δ-topology on L^δ

$$\delta^{\uparrow} = \left[\, \{ \uparrow p \;:\; p \in F(L^\delta) \} \,\right]$$

$$\delta^{\downarrow} = \left[\, \{ \downarrow u \;:\; u \in I(L^\delta) \} \,\right]$$

$$\delta = \delta^{\uparrow} \vee \delta^{\downarrow} = \left[\, \{ [p, u] \;:\; p \in F(L^\delta), u \in I(L^\delta) \} \,\right]$$

(Gehrke and Jónsson, 2004)

The δ topology is used to look at the extension of maps.

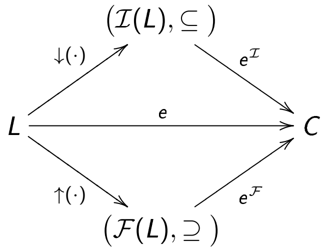
$$\delta = \delta^{\uparrow} \vee \delta^{\downarrow} = \left[\, \{\, [p, u] \;:\; p \in F(L^{\delta}), u \in I(L^{\delta}) \,\}\, \right]$$

Suppose $f : L \longrightarrow M$ and $e : L \hookrightarrow C$. For $x \in C$:

$$f^{\sigma}(x) = \bigvee \left\{ \bigwedge f([p, u] \cap L) \;:\; p \in F(L^{\delta}), u \in I(L^{\delta}), p \leqslant x \leqslant u \right\}$$

$$f^{\pi}(x) = \bigwedge \left\{ \bigvee f([p, u] \cap L) \;:\; p \in F(L^{\delta}), u \in I(L^{\delta}), p \leqslant x \leqslant u \right\}$$

Both f^{σ} and f^{π} extend f, and $f^{\sigma} \leqslant f^{\pi}$.



Lemma: For $e : L \hookrightarrow C$ the following are equivalent:

(i) for all $F \in \mathcal{F}(L), I \in \mathcal{I}(L)$, $\bigwedge e[F] \leqslant \bigvee e[I] \implies F \cap I \neq \emptyset$,

(ii) $\beta : \mathcal{I}(L) \longrightarrow C$ is $(\sigma, \delta^{\uparrow})$-continuous, $\alpha : \mathcal{F}(L) \longrightarrow C$ is $(\sigma^{\partial}, \delta^{\downarrow})$-continuous.

(Vosmaer, 2009)

Theorem:

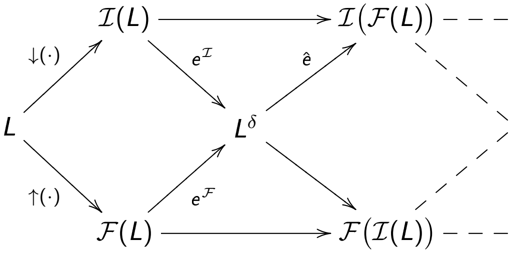
Let $e : L \hookrightarrow C$ be a completion of L. Then (e, C) a canonical extension iff:

(i) $\beta : \mathcal{I}(L) \longrightarrow C$ is $(\sigma, \delta^{\uparrow})$-continuous, $\alpha : \mathcal{F}(L) \longrightarrow C$ is $(\sigma^{\partial}, \delta^{\downarrow})$-continuous,

(ii) $\delta^{\uparrow}$ and $\delta^{\downarrow}$ are both T_0.

(Vosmaer, 2009)

Hierarchy of completions (*Gehrke & Priestley 2008*)



Theorem: The embedding $\hat{e} : L^\delta \hookrightarrow \mathcal{I}(\mathcal{F}(L))$ is a $(\delta^\uparrow, \sigma)$-homeomorphic embedding.

(Vosmaer, 2009)

Alternative statement: the topology $\delta^{\downarrow}$ on L^{δ} is the subspace topology from the Scott topology on $\mathcal{I}(\mathcal{F}(L))$.

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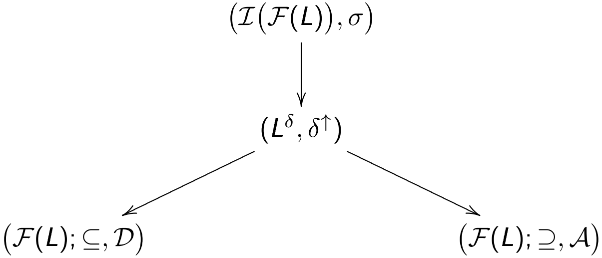

What about the $\delta^{\downarrow}$-topology?

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$$
\begin{array}{ccc}
 & \big(\mathcal{I}(\mathcal{F}(L)), \mathcal{D}\big) & \\
 & \downarrow & \\
 & (L^{\delta}, \delta^{\downarrow}) & \\
 \swarrow & & \searrow \\
\big(\mathcal{F}(L); \subseteq, \sigma\big) & & \big(\mathcal{F}(L); \supseteq, \sigma^{\partial}\big)
\end{array}
$$

Restriction of the δ-topology

Let D be a distributive lattice and $(\mathcal{F}_p(D), \subseteq)$, the prime filters ordered by inclusion. Consider $(\mathcal{F}_p(D), \subseteq)$ as a subset of D^δ under the embedding:

$$F \longmapsto \bigwedge F.$$

Result: Let $\delta_R^\downarrow$ be the $\delta^\downarrow$-topology restricted to $\mathcal{F}_p(D)$. Then $\delta_R^\downarrow = \gamma$, the Stone topology on $\mathcal{F}_p(L)$.

(Gehrke, unpublished)

Now for L an arbitrary lattice, consider $(\mathcal{F}(L), \subseteq)$ with the same embedding into L^δ.

Result: The topology $\delta_R^\downarrow$ on $(\mathcal{F}(L), \subseteq)$ is the Scott topology.

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