Dedekind-MacNeille completions of residuated lattices

Joint work with A. Ciabattoni and K. Terui

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Heying algebras

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A Heyting algebra is a structure $\mathbf{A} = (A, \wedge, \vee, \to, 0, 1)$ where

- $(A, \wedge, \vee, 0, 1)$ is a bounded lattice,
- for all $a, b, c \in A$,

$$a \wedge b \leq c \Leftrightarrow b \leq a \to c \text{ ($\wedge$-residuation)}$$

Heyting algebras provide algebraic semantics for intuitionistic propositional logic. [Eg: topologies, locales/frames.]

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Theorem. [Bezhanishvilli-Harding, '04] The only varieties (i.e., equationally defined classes) of Heyting algebras that are closed under *Dedekind-MacNeille completions* are the trivial, BA and HA.

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Fact: The DM-completion of $\mathbf{A}$ is the unique (up to isomorphism) completion in which $\mathbf{A}$ is both meet dense and join dense. Namely, every element a can be written as

$$a = \bigvee X = \bigwedge Y \quad \text{for some } X, Y \subseteq A.$$

Gödel algebras

A class of ordered algebras is *closed under completions* if every algebra in the class embeds in a complete algebra in the class.

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A *Gödel algebra* is a Heyting algebra that satisfies

$$(x \to y) \vee (y \to x) = 1 \text{ (prelinearity).}$$

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Since totally ordered Heyting algebras are closed under DM-completions we get

$$\mathbf{A} \hookrightarrow \prod_{i \in I} \mathbf{A}_i \hookrightarrow \prod_{i \in I} \overline{\mathbf{A}}_i.$$

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Proposition: Gödel algebras are closed under completions.

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Fact: $A \models (x \to y) \vee (y \to x) = 1$ iff $A \models x \le y$ or $y \le x$ (lin), for $\mathbf{A} \in \mathsf{HA}_{SI}$.

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Proof: Assume $\mathbf{A}$ satisfies (lin').

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$$\bigvee Z \le \bigwedge X \text{ and } \bigvee W \le \bigwedge Y \Longrightarrow \bigvee W \le \bigwedge X \text{ or } \bigvee Z \le \bigwedge Y$$

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This contradicts (lin').

Residuated lattices

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- $(L, \wedge, \vee)$ is a lattice,
- $(L, \cdot, 1)$ is a monoid and
- for all $a, b, c \in L$,

$$ab \leq c \;\Leftrightarrow\; b \leq a\backslash c \;\Leftrightarrow\; a \leq c/b.$$

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Fact. The last condition is equivalent to either one of:

- Multiplication distributes over existing $\bigvee$'s and, for all $a, c \in L$, $\bigvee\{b : ab \leq c\}$ $(=: a\backslash c)$ and $\bigvee\{b : ba \leq c\}$ $(=: c/a)$ exist.

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- (For complete lattices) $\cdot$ distributes over $\bigvee$. [Quantales]
- For all $a, b, c \in L$, $\begin{array}{ll} b \leq a\backslash(ab \vee c) & a \leq (c \vee ab)/b \\ a(a\backslash c \wedge b) \leq c & (a \wedge c/b)b \leq c \end{array}$

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Therefore, the class RL of residuated lattices is an equational class/variety. We write $x \to y$ for $x\backslash y$ and y/x, when they are equal.

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Therefore, the class RL of residuated lattices is an equational class/variety. We write $x \to y$ for $x\backslash y$ and y/x, when they are equal.

We also add in the language a constant 0, for which we stipulate nothing. It allows the definition of negation(s) $\neg x := x \to 0$.

Examples

- Lattice-ordered groups. For $x\backslash y = x^{-1}y$, $y/x = yx^{-1}$.
- (Reducts of) relation algebras. For $x \cdot y = x;y$, $x\backslash y = (x^{\cup}; y^c)^c$, $y/x = (y^c; x^{\cup})^c$, $1 = id$ and $0 = id^c$.
- The powerset $(\mathcal{P}(M), \cap, \cup, \cdot, \backslash, /, \{e\})$ of a monoid $\mathbf{M} = (M, \cdot, e)$, where $X \cdot Y = \{x \cdot y \mid x \in X,\ y \in Y\}$, $X/Y = \{z \in M \mid \{z\} \cdot Y \subseteq X\}$, $Y\backslash Y = \{z \in M \mid Y \cdot \{z\} \subseteq X\}$.
- Ideals of a ring (with 1), where $IJ = \{\sum_{fin} ij \mid i \in I, j \in J\}$ $I/J = \{k \mid kJ \subseteq I\}$, $J\backslash I = \{k \mid Jk \subseteq I\}$, $1 = R$.
- Quantales are (essentially) complete residuated lattices.
- Boolean algebras. $x/y = y\backslash x = y \to x = y^c \vee x$ and $x \cdot y = x \wedge y$.
- MV-algebras. For $x \cdot y = x \odot y$ and $x\backslash y = y/x = \neg(\neg x \odot y)$.
- Models of relevance and of linear logic.

Properties

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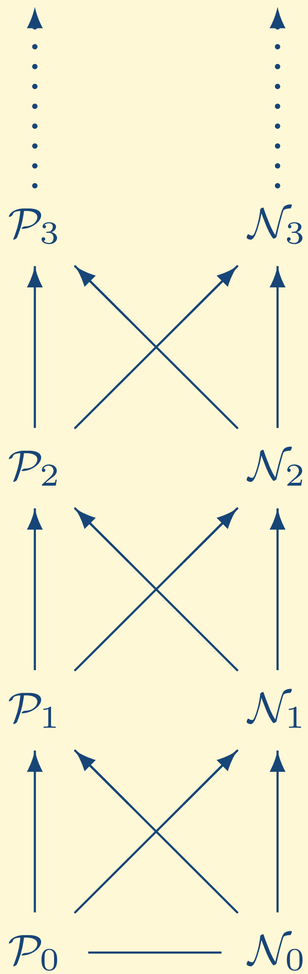
For $\{\vee, \cdot, 1\}$

- $x \cdot 1 = x = 1 \cdot x$
- $x(y \vee z) = xy \vee xz$ and $(y \vee z)x = yx \vee zx$

For $\{\wedge, \backslash, /\}$ (and $\{\vee, \cdot, 1\}$ in the denominator)

- $x\backslash(y/z) = (x\backslash y)/z$
- $x\backslash(y \wedge z) = (x\backslash y) \wedge (x\backslash z)$ and $(y \wedge z)/x = (y/x) \wedge (z/x)$
- $(y \vee z)\backslash x = (y\backslash x) \wedge (z\backslash x)$ and $x/(y \vee z) = (x/y) \wedge (x/z)$
- $(yz)\backslash x = z\backslash(y\backslash x)$ and $x/(zy) = (x/y)/z$
- $1\backslash x = x = x/1$

Term hierarchy



- Polarity $\{\vee, \cdot, 1\}$, $\{\wedge, \backslash, /\}$
- The sets $\mathcal{P}_n, \mathcal{N}_n$ of terms are defined by:
  (0) $\mathcal{P}_0 = \mathcal{N}_0 =$ the set of variables
  (P1) $\mathcal{N}_n \cup \{1\} \subseteq \mathcal{P}_{n+1}$
  (P2) $\alpha, \beta \in \mathcal{P}_{n+1} \quad \Rightarrow \quad \alpha \vee \beta, \alpha \cdot \beta \in \mathcal{P}_{n+1}$
  (N1) $\mathcal{P}_n \cup \{0\} \subseteq \mathcal{N}_{n+1}$
  (N2) $\alpha, \beta \in \mathcal{N}_{n+1} \quad \Rightarrow \quad \alpha \wedge \beta \in \mathcal{N}_{n+1}$
  (N3) $\alpha \in \mathcal{P}_{n+1}, \beta \in \mathcal{N}_{n+1} \quad \Rightarrow \quad \alpha \backslash \beta, \beta / \alpha \in \mathcal{N}_{n+1}$
- $\mathcal{P}_{n+1} = \langle \mathcal{N}_n \rangle_{\bigvee, \prod}$; $\mathcal{N}_{n+1} = \langle \mathcal{P}_n \rangle_{\bigwedge, \mathcal{P}_{n+1} \backslash, / \mathcal{P}_{n+1}}$
- $\mathcal{P}_n \subseteq \mathcal{P}_{n+1}, \mathcal{N}_n \subseteq \mathcal{N}_{n+1}, \bigcup \mathcal{P}_n = \bigcup \mathcal{N}_n = Fm$
- $\mathcal{P}_1$-reduced: $\bigvee \prod p_i$
- $\mathcal{N}_1$-reduced: $\bigwedge (p_1 p_2 \cdots p_n \backslash r / q_1 q_2 \cdots q_m)$

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$\mathcal{N}_2$-normal formulas are of the form $\alpha_1 \cdots \alpha_n \to \beta$ where

- $\beta = 0$ or $\beta_1 \vee \cdots \vee \beta_k$ with each β_i a product of variables
- each α_i is of the form $\bigwedge_{1 \leq j \leq m_i} \gamma_i^j \to \beta_i^j$, where $\beta_i^j = 0$ or is a variable, and γ_i^j is a product of variables.

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For any set E of $\mathcal{N}_2$-equations, the following are equivalent:

- The variety $Mod(E)$ is closed under completions.
- The variety $Mod(E)$ is closed under DM-completions.
- E is equivalent to a set of acyclic equations.
- E is equivalent to a set of analytic equations.

If E implies integrality $x \leq 1$, all the above hold.

Completions

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We restrict to varieties that satisfy $xy = yx$ and $x \leq 1$. (ICRL)

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A *structural clause* is a universal first-order formula of the form:

$$t_1 \leq u_1 \text{ and } \ldots \text{ and } t_m \leq u_m \Longrightarrow t_{m+1} \leq u_{m+1} \text{ or } \ldots \text{ or } t_n \leq u_n$$

where for every $1 \leq i \leq n$, t_i is a product of variables or 1 and u_i is either a variable or 0.

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We restrict to varieties that satisfy $xy = yx$ and $x \leq 1$. (ICRL)

A *structural clause* is a universal first-order formula of the form:

$$t_1 \leq u_1 \text{ and } \ldots \text{ and } t_m \leq u_m \Longrightarrow t_{m+1} \leq u_{m+1} \text{ or } \ldots \text{ or } t_n \leq u_n$$

where for every $1 \leq i \leq n$, t_i is a product of variables or 1 and u_i is either a variable or 0.

Theorem. For SI algebras, each equation in $\mathcal{P}_3$ is equivalent to a finite set of structural clauses.

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Let $L = var\{t_{m+1}, \ldots, t_n\}$ and $R = var\{u_{m+1}, \ldots, u_n\}$. The clause is called *analytic* if it satisfies:

- L and R are disjoint.
- Each variable occurs only once in $t_{m+1}, u_{m+1}, \ldots, t_n, u_n$.
- $var\{t_1, \ldots, t_m\} \subseteq L$ and $var\{u_1, \ldots, u_m\} \subseteq R$.

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Theorem. Analytic clauses are preserved by DM-completions.

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In the absence of commutativity and integrality we need to consider:

1. *iterated conjugates*:

A *conjugate* of a term t is either $\lambda_u(t) = (u\backslash tu) \wedge 1$ or $\rho_u(t) = (ut/u) \wedge 1$ for some term u. We have:

$$\lambda_u(t) \leq 1, \qquad \rho_u(t) \leq 1, \qquad u\lambda_u(t) \leq tu, \qquad \rho_u(t)u \leq ut.$$

Iterated conjugates are compositions of conjugates.

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2. *acyclic clauses*:

A clause

$$t_1 \leq u_1 \text{ and } \ldots \text{ and } t_m \leq u_m \Longrightarrow t_{m+1} \leq u_{m+1} \text{ or } \ldots \text{ or } t_n \leq u_n$$

is called acyclic if there are no directed cycles in the directed graph (G, E), where $G = var\{t_1, u_1, \ldots, t_m, u_m\}$, and $(x, y) \in E$ iff $lxr \leq y$ is a premise.

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