Topological games

Gary Gruenhage
Auburn University

May 31, 2010

The general setup

I'll mostly talk about games of length ω of the following type:

The general setup

I'll mostly talk about games of length ω of the following type:

There are two players, I and II.

The general setup

I'll mostly talk about games of length ω of the following type:
There are two players, I and II.
In the n^{th} round, I chooses a set I_n (in some restricted collection of sets defined by the "rules" of the game). II responds by choosing a set J_n.

The general setup

I'll mostly talk about games of length ω of the following type:

There are two players, I and II.

In the n^{th} round, I chooses a set I_n (in some restricted collection of sets defined by the "rules" of the game). II responds by choosing a set J_n.

I:	I_0		I_1		$\ldots$		I_n		
									$\ldots$
II:		J_0		J_1		$\ldots$		J_n	

The general setup

I'll mostly talk about games of length ω of the following type:

There are two players, I and II.

In the n^{th} round, I chooses a set I_n (in some restricted collection of sets defined by the "rules" of the game). II responds by choosing a set J_n.

$$\begin{array}{llllllll} \text{I:} & I_0 & & I_1 & & \ldots & & I_n & & \\ & & & & & & & & & \ldots \\ \text{II:} & & J_0 & & J_1 & & \ldots & & J_n & \end{array}$$

There is a round for every $n \in \omega$, and then the game is over.

The general setup

I'll mostly talk about games of length ω of the following type:

There are two players, I and II.

In the n^{th} round, I chooses a set I_n (in some restricted collection of sets defined by the "rules" of the game). II responds by choosing a set J_n.

I: I_0 $\quad$ I_1 $\quad$ $\ldots$ $\quad$ I_n $\quad$ $\ldots$

II: $\quad$ J_0 $\quad$ J_1 $\quad$ $\ldots$ $\quad$ J_n

There is a round for every $n \in \omega$, and then the game is over.

I *wins* the game if the sequence

$$I_0, J_0, I_1, J_1, \ldots$$

of plays of the game satisfies a certain condition (e.g., $\bigcap_{n\in\omega} J_n = \emptyset$); otherwise II wins.

The general setup

I'll mostly talk about games of length ω of the following type:
There are two players, I and II.
In the n^{th} round, I chooses a set I_n (in some restricted collection of sets defined by the "rules" of the game). II responds by choosing a set J_n.

I: I_0 $\quad$ I_1 $\quad$ $\ldots$ $\quad$ I_n $\quad$ $\ldots$

II: J_0 $\quad$ J_1 $\quad$ $\ldots$ $\quad$ J_n

There is a round for every $n \in \omega$, and then the game is over.
I *wins* the game if the sequence

$$I_0, J_0, I_1, J_1, \ldots$$

of plays of the game satisfies a certain condition (e.g., $\bigcap_{n\in\omega} J_n = \emptyset$); otherwise II wins.

In a topological game, the sets I_n and J_n of course are topological objects, e.g., points in a space X, closed subsets of a space, an open cover of a space, etc.

A *strategy* for Player I is a function σ whose domain is the set of finite initial segments $I_0, J_0, \ldots, I_n, J_n$ of plays of the game ending with a play by Player II.

A *strategy* for Player I is a function σ whose domain is the set of finite initial segments $I_0, J_0, ..., I_n, J_n$ of plays of the game ending with a play by Player II.

The strategy σ is a *winning strategy for Player I* if I wins every sequence

$$I_0, J_0, I_1, J_1, \ldots$$

of legal plays of the game in which $I_n = \sigma(I_0, J_0, ..., I_{n-1}, J_{n-1})$ for every $n \in \omega$.

A *strategy* for Player I is a function σ whose domain is the set of finite initial segments $I_0, J_0, ..., I_n, J_n$ of plays of the game ending with a play by Player II.

The strategy σ is a *winning strategy for Player I* if I wins every sequence

$$I_0, J_0, I_1, J_1, \ldots$$

of legal plays of the game in which $I_n = \sigma(I_0, J_0, ..., I_{n-1}, J_{n-1})$ for every $n \in \omega$.

Wlog, a strategy for Player I may be considered to be a function whose domain is a the set of finite sequences $J_0, J_1, \ldots$ of plays by Player II, since given a strategy σ as above, and $J_0, ..., J_n$, there is a unique way to fill in the plays $I_0 = \sigma(\emptyset)$, $I_1 = \sigma(I_0, J_0)$, etc. of Player I.

A *strategy* for Player I is a function σ whose domain is the set of finite initial segments $I_0, J_0, ..., I_n, J_n$ of plays of the game ending with a play by Player II.
The strategy σ is a *winning strategy for Player I* if I wins every sequence

$$I_0, J_0, I_1, J_1, \ldots$$

of legal plays of the game in which $I_n = \sigma(I_0, J_0, ..., I_{n-1}, J_{n-1})$ for every $n \in \omega$.

Wlog, a strategy for Player I may be considered to be a function whose domain is a the set of finite sequences $J_0, J_1, \ldots$ of plays by Player II, since given a strategy σ as above, and $J_0, ..., J_n$, there is a unique way to fill in the plays $I_0 = \sigma(\emptyset)$, $I_1 = \sigma(I_0, J_0)$, etc. of Player I.

A (winning) strategy for Player II is defined mutatis mutandis.

A *strategy for Player I* is a function σ whose domain is the set of finite initial segments $I_0, J_0, ..., I_n, J_n$ of plays of the game ending with a play by Player II.

The strategy σ is a *winning strategy for Player I* if I wins every sequence

$$I_0, J_0, I_1, J_1, \ldots$$

of legal plays of the game in which $I_n = \sigma(I_0, J_0, ..., I_{n-1}, J_{n-1})$ for every $n \in \omega$.

Wlog, a strategy for Player I may be considered to be a function whose domain is a the set of finite sequences $J_0, J_1, \ldots$ of plays by Player II, since given a strategy σ as above, and $J_0, ..., J_n$, there is a unique way to fill in the plays $I_0 = \sigma(\emptyset)$, $I_1 = \sigma(I_0, J_0)$, etc. of Player I.

A (winning) strategy for Player II is defined mutatis mutandis.

Obviously, I and II cannot both have a winning strategy, and it is possible that neither has. A game is *determined* if one of the players has a winning strategy, otherwise it is *undetermined*.

Banach-Mazur game BM(X) on X:

Banach-Mazur game

Banach-Mazur game BM(X) on X:

Players E and NE alternately choose nonempty open sets in X:

E: $U_0 \quad U_1$

$\ldots$

NE: $\quad V_0 \quad V_1$

Banach-Mazur game

Banach-Mazur game BM(X) on X:

Players E and NE alternately choose nonempty open sets in X:

E: U_0 U_1

...

NE: V_0 V_1

such that $U_0 \supset V_0 \supset U_1 \supset V_1 \ldots$.

Banach-Mazur game

Banach-Mazur game BM(X) on X:

Players E and NE alternately choose nonempty open sets in X:

E: $U_0 \quad U_1$

$\ldots$

NE: $\quad V_0 \quad V_1$

such that $U_0 \supset V_0 \supset U_1 \supset V_1 \ldots$.

E *wins* if $\bigcap_{n\in\omega} U_n = \emptyset$.

Banach-Mazur game

Banach-Mazur game BM(X) on X:

Players E and NE alternately choose nonempty open sets in X:

E: $U_0 \quad U_1$

$\ldots$

NE: $\quad V_0 \quad V_1$

such that $U_0 \supset V_0 \supset U_1 \supset V_1 \ldots$.

E *wins* if $\bigcap_{n\in\omega} U_n = \emptyset$.

Banach-Mazur game BM(X) on X:

Players E and NE alternately choose nonempty open sets in X:

E: U_0 U_1

...

NE: V_0 V_1

such that $U_0 \supset V_0 \supset U_1 \supset V_1 \ldots$.

E *wins* if $\bigcap_{n\in\omega} U_n = \emptyset$.

Theorem (Oxtoby)

A space X is a Baire space iff E has no winning strategy in BM(X).

Banach-Mazur game

Banach-Mazur game BM(X) on X:

Players E and NE alternately choose nonempty open sets in X:

E: $U_0 \quad U_1$

$\ldots$

NE: $\quad V_0 \quad V_1$

such that $U_0 \supset V_0 \supset U_1 \supset V_1 \ldots$.

E *wins* if $\bigcap_{n\in\omega} U_n = \emptyset$.

Theorem (Oxtoby)

A space X is a Baire space iff E has no winning strategy in BM(X).

If G(X) is a game on X, we'll write "I $\uparrow$ G(X)" to mean "Player I has a winning strategy in G(X)"

Banach-Mazur game

Banach-Mazur game BM(X) on X:

Players E and NE alternately choose nonempty open sets in X:

E: $U_0 \quad U_1$

$\ldots$

NE: $\quad V_0 \quad V_1$

such that $U_0 \supset V_0 \supset U_1 \supset V_1 \ldots$.

E *wins* if $\bigcap_{n\in\omega} U_n = \emptyset$.

Theorem (Oxtoby)

A space X is a Baire space iff E has no winning strategy in BM(X).

If G(X) is a game on X, we'll write "I $\uparrow$ G(X)" to mean "Player I has a winning strategy in G(X)"

So: X is Baire iff E $\not\uparrow$ BM(X)

A space X in which NE $\uparrow$ BM(X) is called *weakly α-favorable*

A space X in which NE $\uparrow$ BM(X) is called *weakly α-favorable*

X completely metrizable (or Čech complete) $\Rightarrow$ NE $\uparrow$ BM(X) $\Rightarrow$ Baire

A space X in which NE $\uparrow$ BM(X) is called *weakly α-favorable*

X completely metrizable (or Čech complete) $\Rightarrow$ NE $\uparrow$ BM(X) $\Rightarrow$ Baire

There are metric Baire spaces X, Y with $X \times Y$ not Baire

A space X in which NE $\uparrow$ BM(X) is called *weakly α-favorable*

X completely metrizable (or Čech complete) $\Rightarrow$ NE $\uparrow$ BM(X) $\Rightarrow$ Baire

There are metric Baire spaces X, Y with $X \times Y$ not Baire

NE $\uparrow$ BM(X_α) for each $\alpha \Rightarrow$ NE $\uparrow$ BM($\Pi_{\alpha\in\kappa} X_\alpha$)

A space X in which NE $\uparrow$ BM(X) is called *weakly α-favorable*

X completely metrizable (or Čech complete) $\Rightarrow$ NE $\uparrow$ BM(X) $\Rightarrow$ Baire

There are metric Baire spaces X, Y with $X \times Y$ not Baire

NE $\uparrow$ BM(X_α) for each $\alpha \Rightarrow$ NE $\uparrow$ BM($\Pi_{\alpha\in\kappa} X_\alpha$)

A space X in which NE $\uparrow$ BM(X) is called *weakly α-favorable*

X completely metrizable (or Čech complete) $\Rightarrow$ NE $\uparrow$ BM(X) $\Rightarrow$ Baire

There are metric Baire spaces X, Y with $X \times Y$ not Baire

NE $\uparrow$ BM(X_α) for each $\alpha \Rightarrow$ NE $\uparrow$ BM($\Pi_{\alpha\in\kappa}X_\alpha$)

Easy: NE $\uparrow$ $BM(X) \Rightarrow X^\kappa$ with box topology Baire $\forall\kappa$

A space X in which NE $\uparrow$ BM(X) is called *weakly α-favorable*

X completely metrizable (or Čech complete) $\Rightarrow$ NE $\uparrow$ BM(X) $\Rightarrow$ Baire

There are metric Baire spaces X, Y with $X \times Y$ not Baire

NE $\uparrow$ BM(X_α) for each $\alpha \Rightarrow$ NE $\uparrow$ BM($\Pi_{\alpha\in\kappa} X_\alpha$)

Easy: NE $\uparrow$ $BM(X) \Rightarrow X^\kappa$ with box topology Baire $\forall\kappa$

Conjecture (Galvin)

Converse is true: NE $\uparrow$ $BM(X) \iff X^\kappa$ with box topology Baire $\forall\kappa$

A space X in which NE $\uparrow$ BM(X) is called *weakly α-favorable*

X completely metrizable (or Čech complete) $\Rightarrow$ NE $\uparrow$ BM(X) $\Rightarrow$ Baire

There are metric Baire spaces X, Y with $X \times Y$ not Baire

NE $\uparrow$ BM(X_α) for each $\alpha \Rightarrow$ NE $\uparrow$ BM($\Pi_{\alpha\in\kappa} X_\alpha$)

Easy: NE $\uparrow$ $BM(X) \Rightarrow X^\kappa$ with box topology Baire $\forall\kappa$

Conjecture (Galvin)

Converse is true: NE $\uparrow$ $BM(X) \iff X^\kappa$ with box topology Baire $\forall\kappa$

Conjecture is consistent if there is a proper class of measurables;

A space X in which NE $\uparrow$ BM(X) is called *weakly α-favorable*

X completely metrizable (or Čech complete) $\Rightarrow$ NE $\uparrow$ BM(X) $\Rightarrow$ Baire

There are metric Baire spaces X, Y with $X \times Y$ not Baire

NE $\uparrow$ BM(X_α) for each $\alpha \Rightarrow$ NE $\uparrow$ BM($\Pi_{\alpha\in\kappa} X_\alpha$)

Easy: NE $\uparrow$ $BM(X) \Rightarrow X^\kappa$ with box topology Baire $\forall\kappa$

Conjecture (Galvin)

Converse is true: NE $\uparrow$ $BM(X) \iff X^\kappa$ with box topology Baire $\forall\kappa$

Conjecture is consistent if there is a proper class of measurables; true in ZFC?

Stationary strategies

A strategy of a player is *stationary* if that players next move depends only on the previous move of his opponent.

Stationary strategies

A strategy of a player is *stationary* if that players next move depends only on the previous move of his opponent.

Galvin and Telgarsky obtained a result which says that for games in a certain class, call it $STAT$, if Player I has a winning strategy, then she has a stationary winning strategy.

Stationary strategies

A strategy of a player is *stationary* if that players next move depends only on the previous move of his opponent.

Galvin and Telgarsky obtained a result which says that for games in a certain class, call it *STAT*, if Player I has a winning strategy, then she has a stationary winning strategy.

BM(X) in this class. If E has a winning strategy in the BM(X), then E has a stationary winning strategy.

Idea of proof:

Idea of proof:
Let σ be a winning strategy for E.

Idea of proof:

Let σ be a winning strategy for E.

Fix a well-order $\prec$ of the collection of all open sets.

Idea of proof:

Let σ be a winning strategy for E.

Fix a well-order $\prec$ of the collection of all open sets.

Suppose NE plays $V_n = V$ in round n.

Idea of proof:

Let σ be a winning strategy for E.

Fix a well-order $\prec$ of the collection of all open sets.

Suppose NE plays $V_n = V$ in round n.

Look at the set $\mathcal{P}(V)$ of all partial legal plays $(U_0, V_0, ..., U_k, V_k)$ of the game with E using σ and with $V_k = V$.

Idea of proof:

Let σ be a winning strategy for E.

Fix a well-order $\prec$ of the collection of all open sets.

Suppose NE plays $V_n = V$ in round n.
Look at the set $\mathcal{P}(V)$ of all partial legal plays $(U_0, V_0, ..., U_k, V_k)$ of the game with E using σ and with $V_k = V$.
One may check:
(i)The lexicographic order on $\mathcal{P}(V)$ is a well-order.

Idea of proof:

Let σ be a winning strategy for E.

Fix a well-order $\prec$ of the collection of all open sets.

Suppose NE plays $V_n = V$ in round n.
Look at the set $\mathcal{P}(V)$ of all partial legal plays $(U_0, V_0, ..., U_k, V_k)$ of the game with E using σ and with $V_k = V$.
One may check:
(i)The lexicographic order on $\mathcal{P}(V)$ is a well-order.
(ii) Given $V_n = V$ is NE's move in round n, let
$\tau(V) = \sigma(V_0, V_1, ..., V_k = V)$, where $(V_0, V_1, ..., V_k = V)$ is the least element of $\mathcal{P}(V)$ ending in V. Then τ defines a stationary winning strategy for E.

Example (Debs)

$\exists X$ with NE$\uparrow$BM(X) but NE has no stationary winning strategy.

Example (Debs)

$\exists X$ with NE$\uparrow$BM(X) but NE has no stationary winning strategy.

NE has strategy depending on last two moves of E.

Example (Debs)

$\exists X$ with NE$\uparrow$BM(X) but NE has no stationary winning strategy.

NE has strategy depending on last two moves of E.
Open: must they all?

Example (Debs)

$\exists X$ with NE$\uparrow$BM(X) but NE has no stationary winning strategy.

NE has strategy depending on last two moves of E.
Open: must they all?

Question (Telgarsky)

$\exists X$ with NE having winning strategy based on last 3 moves of E but not last 2?

Example (Debs)

$\exists X$ with NE↑BM(X) but NE has no stationary winning strategy.

NE has strategy depending on last two moves of E.
Open: must they all?

Question (Telgarsky)

$\exists X$ with NE having winning strategy based on last 3 moves of E but not last 2?

Remark: Galvin and Telgarsky, Debs: NE ↑ BM(X) ⇒ NE has winning strategy based on last move of opponent and his own last move.

Telgársky's game

Let X be a space, and let $\mathbb{K}$ be a closed hereditary class of spaces.

Let X be a space, and let $\mathbb{K}$ be a closed hereditary class of spaces. We define the game $G(\mathbb{K}, X)$. There are two players, I and II.

Telgársky's game

Let X be a space, and let $\mathbb{K}$ be a closed hereditary class of spaces. We define the game $G(\mathbb{K}, X)$. There are two players, I and II. Player I begins by choosing a nonempty closed subset A_0 of X such that $A_0 \in \mathbb{K}$.

Telgársky's game

Let X be a space, and let $\mathbb{K}$ be a closed hereditary class of spaces.
We define the game $G(\mathbb{K}, X)$. There are two players, I and II.
Player I begins by choosing a nonempty closed subset A_0 of X such that $A_0 \in \mathbb{K}$.
II responds by choosing a closed set $B_0 \subset X \setminus A_0$.

Telgársky's game

Let X be a space, and let $\mathbb{K}$ be a closed hereditary class of spaces.
We define the game $G(\mathbb{K}, X)$. There are two players, I and II.
Player I begins by choosing a nonempty closed subset A_0 of X such that $A_0 \in \mathbb{K}$.
II responds by choosing a closed set $B_0 \subset X \setminus A_0$.
Player I then chooses a nonempty closed $A_1 \subset B_0$ with $A_1 \in \mathbb{K}$.

Telgársky's game

Let X be a space, and let $\mathbb{K}$ be a closed hereditary class of spaces.
We define the game $G(\mathbb{K}, X)$. There are two players, I and II.
Player I begins by choosing a nonempty closed subset A_0 of X such that $A_0 \in \mathbb{K}$.
II responds by choosing a closed set $B_0 \subset X \setminus A_0$.
Player I then chooses a nonempty closed $A_1 \subset B_0$ with $A_1 \in \mathbb{K}$.
II chooses a closed $B_1 \subset B_0 \setminus A_1$
etc.

I: $A_0 \in \mathbb{K}$ $\quad A_1 \subset B_0, A_1 \in \mathbb{K}$ $\quad \ldots$ $\quad A_n \subset B_{n-1}, A_n \in \mathbb{K}$

II: $B_0 \subset X \setminus A_0$ $\quad B_1 \subset B_0 \setminus A_1$ $\quad \ldots$ $\quad B_n \subset B_{n-1} \setminus A_n$

I: $A_0 \in \mathbb{K}$ $\quad$ $A_1 \subset B_0, A_1 \in \mathbb{K}$ $\quad$ ... $\quad$ $A_n \subset B_{n-1}, A_n \in \mathbb{K}$

II: $B_0 \subset X \setminus A_0$ $\quad$ $B_1 \subset B_0 \setminus A_1$ $\quad$ ... $\quad$ $B_n \subset B_{n-1} \setminus A_n$

We say I *wins* the game if $\bigcap_{n\in\omega} B_n = \emptyset$; otherwise II wins.

I: $A_0 \in \mathbb{K}$ $\quad A_1 \subset B_0, A_1 \in \mathbb{K}$ $\quad \ldots$ $\quad A_n \subset B_{n-1}, A_n \in \mathbb{K}$

II: $B_0 \subset X \setminus A_0$ $\quad B_1 \subset B_0 \setminus A_1$ $\quad \ldots$ $\quad B_n \subset B_{n-1} \setminus A_n$

We say I *wins* the game if $\bigcap_{n\in\omega} B_n = \emptyset$; otherwise II wins.

The space X is said to be $\mathbb{K}$-*like* if Player I has a winning strategy in $G(\mathbb{K}, X)$ (i.e., if I $\uparrow G(\mathbb{K}, X)$).

I: $A_0 \in \mathbb{K}$ $\quad A_1 \subset B_0, A_1 \in \mathbb{K}$ $\quad \ldots$ $\quad A_n \subset B_{n-1}, A_n \in \mathbb{K}$

II: $B_0 \subset X \setminus A_0$ $\quad B_1 \subset B_0 \setminus A_1$ $\quad \ldots$ $\quad B_n \subset B_{n-1} \setminus A_n$

We say I *wins* the game if $\bigcap_{n\in\omega} B_n = \emptyset$; otherwise II wins.

The space X is said to be *$\mathbb{K}$-like* if Player I has a winning strategy in $G(\mathbb{K}, X)$ (i.e., if I $\uparrow G(\mathbb{K}, X)$).

Trivially, $\mathbb{K}$ is contained in the class of $\mathbb{K}$-like spaces.

Let $\mathbb{DC}$ be the class of all topological sums (discrete unions) of compact sets.

Let $\mathbb{DC}$ be the class of all topological sums (discrete unions) of compact sets.

If $X \in \mathbb{DC}$, then $X \times Y$ is paracompact $\forall$ paracompact Y

Let $\mathbb{DC}$ be the class of all topological sums (discrete unions) of compact sets.

If $X \in \mathbb{DC}$, then $X \times Y$ is paracompact $\forall$ paracompact Y (because compact $\times$ paracompact is paracompact)

Let $\mathbb{DC}$ be the class of all topological sums (discrete unions) of compact sets.

If $X \in \mathbb{DC}$, then $X \times Y$ is paracompact $\forall$ paracompact Y (because compact $\times$ paracompact is paracompact)

Theorem (Telgarsky)

If X is paracompact and $\mathbb{DC}$-like, then $X \times Y$ is paracompact for all paracompact spaces Y.

Let $\mathbb{DC}$ be the class of all topological sums (discrete unions) of compact sets.

If $X \in \mathbb{DC}$, then $X \times Y$ is paracompact $\forall$ paracompact Y (because compact $\times$ paracompact is paracompact)

Theorem (Telgarsky)

If X is paracompact and $\mathbb{DC}$-like, then $X \times Y$ is paracompact for all paracompact spaces Y.

(Sub)paracompact scattered spaces, more generally $\mathbb{C}$-scattered (every closed subspace has a point of local compactness), and spaces with a σ-closure-preserving cover by compact sets, are $\mathbb{DC}$-like.

Telgarsky's Conjecture

$X \times Y$ is paracompact $\forall$ paracompact Y iff I has a winning strategy in $G(\mathbb{DC}, X)$ (i.e., X is $\mathbb{DC}$-like)

Telgarsky's Conjecture

$X \times Y$ is paracompact $\forall$ paracompact Y iff I has a winning strategy in $G(\mathbb{DC}, X)$ (i.e., X is $\mathbb{DC}$-like)

Theorem(Alster, 2006)

Telgärsky's Conjecture holds if X has a base of cardinality $\leq \aleph_1$

Telgärsky's game is in STAT. Player I has a winning strategy iff she has a stationary winning strategy.

Telgärsky's game is in STAT. Player I has a winning strategy iff she has a stationary winning strategy.

Thus, X is $\mathbb{K}$-like iff there is a function $\sigma : \mathcal{C}(X) \to \mathbb{K}$, where $\mathcal{C}(X)$ is the collection of nonempty closed subsets of X, such that

Telgärsky's game is in STAT. Player I has a winning strategy iff she has a stationary winning strategy.

Thus, X is $\mathbb{K}$-like iff there is a function $\sigma : \mathcal{C}(X) \to \mathbb{K}$, where $\mathcal{C}(X)$ is the collection of nonempty closed subsets of X, such that

1. $\sigma(C) \subset C$;

Telgärsky's game is in STAT. Player I has a winning strategy iff she has a stationary winning strategy.

Thus, X is $\mathbb{K}$-like iff there is a function $\sigma : \mathcal{C}(X) \to \mathbb{K}$, where $\mathcal{C}(X)$ is the collection of nonempty closed subsets of X, such that

1. $\sigma(C) \subset C$;
2. $\sigma(C) \in \mathbb{K}$;

Telgärsky's game is in STAT. Player I has a winning strategy iff she has a stationary winning strategy.

Thus, X is $\mathbb{K}$-like iff there is a function $\sigma : \mathcal{C}(X) \to \mathbb{K}$, where $\mathcal{C}(X)$ is the collection of nonempty closed subsets of X, such that

1. $\sigma(C) \subset C$;
2. $\sigma(C) \in \mathbb{K}$;
3. If
$$X = B_{-1} \supset B_0 \supset B_1 \cdots \supset B_n \supset \dots$$
is a decreasing sequence of closed sets such that for each $n \in \omega$, $B_n \cap \sigma(B_{n-1}) = \emptyset$, then $\bigcap_{n\in\omega} B_n = \emptyset$.

D-spaces

A space X is a *D-space* if, given an open nbhd $N(x)$ for each $x \in X$, there is a closed discrete $D \subset X$ such that $N(D) = \{N(x) : x \in D\}$ covers X.

D-spaces

A space X is a *D-space* if, given an open nbhd $N(x)$ for each $x \in X$, there is a closed discrete $D \subset X$ such that $N(D) = \{N(x) : x \in D\}$ covers X.

Compact or σ-compact implies D.

D-spaces

A space X is a *D-space* if, given an open nbhd $N(x)$ for each $x \in X$, there is a closed discrete $D \subset X$ such that $N(D) = \{N(x) : x \in D\}$ covers X.

Compact or σ-compact implies D.

Open question:

Do any of the other standard covering properties (e.g., (Lindelöf, paracompact, metacompact, submetacompact,...) imply D?

X is said to be *Menger* if, given open covers $\mathcal{U}_0, \mathcal{U}_1, \ldots$, there are finite $\mathcal{F}_n \subset \mathcal{U}_n$ such that $\bigcup_{n \in \omega} \mathcal{F}_n$ covers X.

X is said to be *Menger* if, given open covers $\mathcal{U}_0, \mathcal{U}_1, \ldots$, there are finite $\mathcal{F}_n \subset \mathcal{U}_n$ such that $\bigcup_{n\in\omega} \mathcal{F}_n$ covers X.

σ-compact $\Rightarrow$ Menger $\Rightarrow$ Lindelöf

X is said to be *Menger* if, given open covers $\mathcal{U}_0, \mathcal{U}_1, \ldots$, there are finite $\mathcal{F}_n \subset \mathcal{U}_n$ such that $\bigcup_{n\in\omega} \mathcal{F}_n$ covers X.

σ-compact $\Rightarrow$ Menger $\Rightarrow$ Lindelöf

Irrationals are not Menger.

X is said to be *Menger* if, given open covers $\mathcal{U}_0, \mathcal{U}_1, \ldots$, there are finite $\mathcal{F}_n \subset \mathcal{U}_n$ such that $\bigcup_{n\in\omega} \mathcal{F}_n$ covers X.

σ-compact $\Rightarrow$ Menger $\Rightarrow$ Lindelöf

Irrationals are not Menger.

$MA(\kappa) \Rightarrow$ any $X \subset \mathbb{R}$ with $|X| \leq \kappa$ is Menger.

X is said to be *Menger* if, given open covers $\mathcal{U}_0, \mathcal{U}_1, \ldots$, there are finite $\mathcal{F}_n \subset \mathcal{U}_n$ such that $\bigcup_{n\in\omega} \mathcal{F}_n$ covers X.

σ-compact $\Rightarrow$ Menger $\Rightarrow$ Lindelöf

Irrationals are not Menger.

$MA(\kappa) \Rightarrow$ any $X \subset \mathbb{R}$ with $|X| \leq \kappa$ is Menger.

Hurewicz: analytic + Menger $\Rightarrow$ σ-compact

X is said to be *Menger* if, given open covers $\mathcal{U}_0, \mathcal{U}_1, \ldots$, there are finite $\mathcal{F}_n \subset \mathcal{U}_n$ such that $\bigcup_{n\in\omega} \mathcal{F}_n$ covers X.

σ-compact $\Rightarrow$ Menger $\Rightarrow$ Lindelöf

Irrationals are not Menger.

$MA(\kappa) \Rightarrow$ any $X \subset \mathbb{R}$ with $|X| \leq \kappa$ is Menger.

Hurewicz: analytic $+$ Menger $\Rightarrow$ σ-compact

(Fremlin-Miller)ZFC$\Rightarrow$ $\exists$ non-σ-compact Menger $X \subset \mathbb{R}$

X is said to be *Menger* if, given open covers $\mathcal{U}_0, \mathcal{U}_1, \ldots$, there are finite $\mathcal{F}_n \subset \mathcal{U}_n$ such that $\bigcup_{n \in \omega} \mathcal{F}_n$ covers X.

σ-compact $\Rightarrow$ Menger $\Rightarrow$ Lindelöf

Irrationals are not Menger.

$MA(\kappa) \Rightarrow$ any $X \subset \mathbb{R}$ with $|X| \leq \kappa$ is Menger.

Hurewicz: analytic $+$ Menger $\Rightarrow$ σ-compact

(Fremlin-Miller)ZFC$\Rightarrow \exists$ non-σ-compact Menger $X \subset \mathbb{R}$

Theorem(Aurichi)

Menger spaces are D-spaces.

X is said to be *Menger* if, given open covers $\mathcal{U}_0, \mathcal{U}_1, \ldots$, there are finite $\mathcal{F}_n \subset \mathcal{U}_n$ such that $\bigcup_{n\in\omega} \mathcal{F}_n$ covers X.

σ-compact $\Rightarrow$ Menger $\Rightarrow$ Lindelöf

Irrationals are not Menger.

$MA(\kappa) \Rightarrow$ any $X \subset \mathbb{R}$ with $|X| \leq \kappa$ is Menger.

Hurewicz: analytic + Menger $\Rightarrow$ σ-compact

(Fremlin-Miller)ZFC$\Rightarrow \exists$ non-σ-compact Menger $X \subset \mathbb{R}$

Theorem(Aurichi)

Menger spaces are D-spaces.

Not clear how to do a direct proof. A game characterization of Menger, due to Hurewicz, provides an easy proof.

Menger game M(X):

Menger game M(X):

In round n, Player I chooses open cover $\mathcal{U}_n$ of X.

Menger game M(X):
In round n, Player I chooses open cover $\mathcal{U}_n$ of X.
II responds with finite $\mathcal{F}_n \subset \mathcal{U}_n$.

Menger game M(X):

In round n, Player I chooses open cover $\mathcal{U}_n$ of X.

II responds with finite $\mathcal{F}_n \subset \mathcal{U}_n$.

II *wins* if $\bigcup_{n\in\omega} \mathcal{F}_n$ covers X.

Menger game M(X):

In round n, Player I chooses open cover $\mathcal{U}_n$ of X.
II responds with finite $\mathcal{F}_n \subset \mathcal{U}_n$.

II *wins* if $\bigcup_{n\in\omega} \mathcal{F}_n$ covers X.
Equivalently, I's cover is closed under finite unions, and II chooses $U_n \in \mathcal{U}_n$. II wins if $\bigcup_{n\in\omega} U_n = X$.

Menger game M(X):
In round n, Player I chooses open cover $\mathcal{U}_n$ of X.
II responds with finite $\mathcal{F}_n \subset \mathcal{U}_n$.

II *wins* if $\bigcup_{n\in\omega} \mathcal{F}_n$ covers X.
Equivalently, I's cover is closed under finite unions, and II chooses $U_n \in \mathcal{U}_n$. II wins if $\bigcup_{n\in\omega} U_n = X$.

Easy: I $\not\uparrow$ M(X)$\Rightarrow$ X Menger

Menger game M(X):

In round n, Player I chooses open cover $\mathcal{U}_n$ of X.
II responds with finite $\mathcal{F}_n \subset \mathcal{U}_n$.

II *wins* if $\bigcup_{n\in\omega} \mathcal{F}_n$ covers X.
Equivalently, I's cover is closed under finite unions, and II chooses $U_n \in \mathcal{U}_n$. II wins if $\bigcup_{n\in\omega} U_n = X$.

Easy: I $\not\uparrow$ M(X)$\Rightarrow$ X Menger

Theorem (Hurewicz)

X is Menger iff I $\not\uparrow$ M(X)

Proof of Aurichi's theorem

Proof of Aurichi's theorem

Assume X Menger.

Proof of Aurichi's theorem

Assume X Menger.

Let N be a neighborhood assignment on X.

Proof of Aurichi's theorem

Assume X Menger.

Let N be a neighborhood assignment on X.

Let Player I's first play be $\{N(x) : x \in X\}$.

Proof of Aurichi's theorem

Assume X Menger.

Let N be a neighborhood assignment on X.

Let Player I's first play be $\{N(x) : x \in X\}$.

Player II responds with $\{N(x) : x \in F_0\}$, $F_0 \in [X]^{<\omega}$.

Proof of Aurichi's theorem

Assume X Menger.

Let N be a neighborhood assignment on X.

Let Player I's first play be $\{N(x) : x \in X\}$.

Player II responds with $\{N(x) : x \in F_0\}$, $F_0 \in [X]^{<\omega}$.

Let $V_0 = \cup\{N(x) : x \in F_0\}$, and let I then play

$$\{V_0 \cup N(x) : x \in X \setminus V_0\}.$$

Proof of Aurichi's theorem

Assume X Menger.

Let N be a neighborhood assignment on X.

Let Player I's first play be $\{N(x) : x \in X\}$.

Player II responds with $\{N(x) : x \in F_0\}$, $F_0 \in [X]^{<\omega}$.

Let $V_0 = \cup\{N(x) : x \in F_0\}$, and let I then play

$$\{V_0 \cup N(x) : x \in X \setminus V_0\}.$$

Then similarly, if II's reply is $\{V_0 \cup N(x) : x \in F_1\}$, where $F_1 \in [X \setminus V_0]^{<\omega}$,

Proof of Aurichi's theorem

Assume X Menger.

Let N be a neighborhood assignment on X.

Let Player I's first play be $\{N(x) : x \in X\}$.

Player II responds with $\{N(x) : x \in F_0\}$, $F_0 \in [X]^{<\omega}$.

Let $V_0 = \cup\{N(x) : x \in F_0\}$, and let I then play

$$\{V_0 \cup N(x) : x \in X \setminus V_0\}.$$

Then similarly, if II's reply is $\{V_0 \cup N(x) : x \in F_1\}$, where $F_1 \in [X \setminus V_0]^{<\omega}$, let $V_1 = V_0 \cup \bigcup\{N(x) : x \in F_1\}$ and let I play

$$\{V_1 \cup N(x) : x \in X \setminus V_1\},$$

Proof of Aurichi's theorem
Assume X Menger.
Let N be a neighborhood assignment on X.
Let Player I's first play be $\{N(x) : x \in X\}$.
Player II responds with $\{N(x) : x \in F_0\}$, $F_0 \in [X]^{<\omega}$.
Let $V_0 = \cup\{N(x) : x \in F_0\}$, and let I then play

$$\{V_0 \cup N(x) : x \in X \setminus V_0\}.$$

Then similarly, if II's reply is $\{V_0 \cup N(x) : x \in F_1\}$, where $F_1 \in [X \setminus V_0]^{<\omega}$, let $V_1 = V_0 \cup \bigcup\{N(x) : x \in F_1\}$ and let I play

$$\{V_1 \cup N(x) : x \in X \setminus V_1\},$$

and so on.

Proof of Aurichi's theorem

Assume X Menger.

Let N be a neighborhood assignment on X.

Let Player I's first play be $\{N(x) : x \in X\}$.

Player II responds with $\{N(x) : x \in F_0\}$, $F_0 \in [X]^{<\omega}$.

Let $V_0 = \cup\{N(x) : x \in F_0\}$, and let I then play

$$\{V_0 \cup N(x) : x \in X \setminus V_0\}.$$

Then similarly, if II's reply is $\{V_0 \cup N(x) : x \in F_1\}$, where $F_1 \in [X \setminus V_0]^{<\omega}$, let $V_1 = V_0 \cup \bigcup\{N(x) : x \in F_1\}$ and let I play

$$\{V_1 \cup N(x) : x \in X \setminus V_1\},$$

and so on.

This defines a strategy for Player I.

X is Menger, so this can't be a winning strategy.

X is Menger, so this can't be a winning strategy.

Therefore there is some play of the game with I using this strategy such that, if $F_0, F_1, \ldots$ code the plays of II, then $X = \bigcup_{n\in\omega} V_n = \bigcup\{N(x) : x \in \bigcup_{n\in\omega} F_n\}$.

X is Menger, so this can't be a winning strategy.

Therefore there is some play of the game with I using this strategy such that, if $F_0, F_1, \ldots$ code the plays of II, then $X = \bigcup_{n\in\omega} V_n = \bigcup\{N(x) : x \in \bigcup_{n\in\omega} F_n\}$.

Let $D = \bigcup_{n\in\omega} F_n$. Then $N(D)$ covers X.

X is Menger, so this can't be a winning strategy.

Therefore there is some play of the game with I using this strategy such that, if $F_0, F_1, \ldots$ code the plays of II, then $X = \bigcup_{n\in\omega} V_n = \bigcup\{N(x) : x \in \bigcup_{n\in\omega} F_n\}$.

Let $D = \bigcup_{n\in\omega} F_n$. Then $N(D)$ covers X.
Since for each n, we have $F_n \subset V_n$ and $F_{n+1} \cap V_n = \emptyset$, it is easy to check that D is a closed discrete subset of X. Hence X is a D-space. □

Proof of Hurewicz's theorem

To show: Menger $\Rightarrow$ I $\not\uparrow$ M(X)

Proof of Hurewicz's theorem

To show: Menger $\Rightarrow$ I $\not\uparrow$ M(X)
Suppose X Menger, and consider any fixed strategy by I. We show it can be defeated.

Proof of Hurewicz's theorem

To show: Menger $\Rightarrow$ I $\not\uparrow$ M(X)

Suppose X Menger, and consider any fixed strategy by I. We show it can be defeated.

X is Lindelöf, so wlog, I always chooses a countable open cover.

Proof of Hurewicz's theorem

To show: Menger $\Rightarrow$ I $\uparrow\!\!\!/$ M(X)
Suppose X Menger, and consider any fixed strategy by I. We show it can be defeated.
X is Lindelöf, so wlog, I always chooses a countable open cover.
Since II is choosing a finite subcollection, it doesn't hurt I to be restricted to increasing open covers.

Proof of Hurewicz's theorem

To show: Menger $\Rightarrow$ I $\not\uparrow$ M(X)
Suppose X Menger, and consider any fixed strategy by I. We show it can be defeated.
X is Lindelöf, so wlog, I always chooses a countable open cover.
Since II is choosing a finite subcollection, it doesn't hurt I to be restricted to increasing open covers.
In this case, it doesn't harm II to be restricted to choosing a single element of the cover.

Proof of Hurewicz's theorem

To show: Menger $\Rightarrow$ I $\not\uparrow$ M(X)
Suppose X Menger, and consider any fixed strategy by I. We show it can be defeated.
X is Lindelöf, so wlog, I always chooses a countable open cover.
Since II is choosing a finite subcollection, it doesn't hurt I to be restricted to increasing open covers.
In this case, it doesn't harm II to be restricted to choosing a single element of the cover.
Finally, if II chooses U, I may as well make the first member of the increasing open cover that is his response contain U.

Proof of Hurewicz's theorem

To show: Menger $\Rightarrow$ I $\not\uparrow$ M(X)

Suppose X Menger, and consider any fixed strategy by I. We show it can be defeated.

X is Lindelöf, so wlog, I always chooses a countable open cover.

Since II is choosing a finite subcollection, it doesn't hurt I to be restricted to increasing open covers.

In this case, it doesn't harm II to be restricted to choosing a single element of the cover.

Finally, if II chooses U, I may as well make the first member of the increasing open cover that is his response contain U.

To summarize: I chooses countable increasing open cover, each member of which contains II's previous move. II chooses a member of I's cover.

Proof of Hurewicz's theorem

To show: Menger $\Rightarrow$ I $\not\uparrow$ M(X)

Suppose X Menger, and consider any fixed strategy by I. We show it can be defeated.

X is Lindelöf, so wlog, I always chooses a countable open cover.

Since II is choosing a finite subcollection, it doesn't hurt I to be restricted to increasing open covers.

In this case, it doesn't harm II to be restricted to choosing a single element of the cover.

Finally, if II chooses U, I may as well make the first member of the increasing open cover that is his response contain U.

To summarize: I chooses countable increasing open cover, each member of which contains II's previous move. II chooses a member of I's cover. Want to show that II can defeat I's strategy.

$$\{U_n\}_n$$

$$\{U_{0m}\}_m \qquad \{U_{1m}\}_m \qquad \ldots \qquad \{U_{nm}\}_m \qquad \ldots$$

$$\{U_{n0k}\}_k \quad \{U_{n1k}\}_k \quad \ldots \quad \{U_{nmk}\}_k$$

$$\{U_n\}_n$$

$$\{U_{0m}\}_m \qquad \{U_{1m}\}_m \qquad \ldots \qquad \{U_{nm}\}_m \qquad \ldots$$

$$\{U_{n0k}\}_k \quad \{U_{n1k}\}_k \quad \ldots \quad \{U_{nmk}\}_k$$

Let $\{U_n\}_n$ be I's first move using the strategy.

$$\{U_n\}_n$$

$$\{U_{0m}\}_m \qquad \{U_{1m}\}_m \qquad \ldots \qquad \{U_{nm}\}_m \qquad \ldots$$

$$\{U_{n0k}\}_k \quad \{U_{n1k}\}_k \quad \ldots \quad \{U_{nmk}\}_k$$

Let $\{U_n\}_n$ be I's first move using the strategy.
If II responds with U_n, let $\{U_{nm}\}_m$ be I's next move.

$$\{U_n\}_n$$

$$\{U_{0m}\}_m \qquad \{U_{1m}\}_m \qquad \ldots \qquad \{U_{nm}\}_m \qquad \ldots$$

$$\{U_{n0k}\}_k \quad \{U_{n1k}\}_k \quad \ldots \quad \{U_{nmk}\}_k$$

Let $\{U_n\}_n$ be I's first move using the strategy.
If II responds with U_n, let $\{U_{nm}\}_m$ be I's next move.
Then if II plays U_{nm}, let $\{U_{nmk}\}_k$ be I's reply.

$$\{U_n\}_n$$

$$\{U_{0m}\}_m \qquad \{U_{1m}\}_m \qquad \dots \qquad \{U_{nm}\}_m \qquad \dots$$

$$\{U_{n0k}\}_k \quad \{U_{n1k}\}_k \quad \dots \{U_{nmk}\}_k$$

Let $\{U_n\}_n$ be I's first move using the strategy.
If II responds with U_n, let $\{U_{nm}\}_m$ be I's next move.
Then if II plays U_{nm}, let $\{U_{nmk}\}_k$ be I's reply.

In this way we define a "game tree" $\{U_\sigma\}_{\sigma \in \omega^{<\omega}}$. (Let $U_\emptyset = \emptyset$.)
We need to show that there is a play of the game, i.e., a branch of the game tree, for which the corresponding open sets cover.

$$\{U_n\}_n$$

$$\{U_{0m}\}_m \qquad \{U_{1m}\}_m \qquad \dots \qquad \{U_{nm}\}_m \qquad \dots$$

$$\{U_{n0k}\}_k \quad \{U_{n1k}\}_k \quad \dots \quad \{U_{nmk}\}_k$$

Let $\{U_n\}_n$ be I's first move using the strategy.
If II responds with U_n, let $\{U_{nm}\}_m$ be I's next move.
Then if II plays U_{nm}, let $\{U_{nmk}\}_k$ be I's reply.

In this way we define a "game tree" $\{U_\sigma\}_{\sigma \in \omega^{<\omega}}$. (Let $U_\emptyset = \emptyset$.)
We need to show that there is a play of the game, i.e., a branch of the game tree, for which the corresponding open sets cover.
That is, we want $f : \omega \to \omega$ such that $X = \bigcup_{n \in \omega} U_{f \restriction n}$.

$$\{U_n\}_n$$

$$\{U_{0m}\}_m \qquad \{U_{1m}\}_m \qquad \ldots \qquad \{U_{nm}\}_m \qquad \ldots$$

$$\{U_{n0k}\}_k \quad \{U_{n1k}\}_k \quad \ldots \quad \{U_{nmk}\}_k$$

Let $\{U_n\}_n$ be I's first move using the strategy.
If II responds with U_n, let $\{U_{nm}\}_m$ be I's next move.
Then if II plays U_{nm}, let $\{U_{nmk}\}_k$ be I's reply.

In this way we define a "game tree" $\{U_\sigma\}_{\sigma\in\omega^{<\omega}}$. (Let $U_\emptyset = \emptyset$.)
We need to show that there is a play of the game, i.e., a branch of the game tree, for which the corresponding open sets cover.
That is, we want $f : \omega \to \omega$ such that $X = \bigcup_{n\in\omega} U_{f\restriction n}$.
A naive idea is to apply the Menger property to the countably many covers $\{U_{\sigma^\frown n}\}_n$. There is a choice of one member of each that covers. But this doesn't get you a branch that covers!

Instead, we use the game tree to define covers $\{V_k^n\}_k$ as follows.

Instead, we use the game tree to define covers $\{V_k^n\}_k$ as follows.
Let $V_k^0 = U_k$; then $\{V_k^0\}_k$ is increasing open cover

Instead, we use the game tree to define covers $\{V_k^n\}_k$ as follows.
Let $V_k^0 = U_k$; then $\{V_k^0\}_k$ is increasing open cover
Next let

$$V_k^1 = U_k \cap U_{0k} \cap U_{1k} \cap \ldots \cap U_{k-1,k}.$$

Instead, we use the game tree to define covers $\{V_k^n\}_k$ as follows.
Let $V_k^0 = U_k$; then $\{V_k^0\}_k$ is increasing open cover
Next let

$$V_k^1 = U_k \cap U_{0k} \cap U_{1k} \cap \ldots \cap U_{k-1,k}.$$

Since we have assumed $U_{ik} \supset U_i$, note that $V_k^1 = \bigcap_{\sigma \in \omega^{\leq 1}} U_{\sigma k}$, i.e.,
$V_k^1 =$ intersection of all k^{th} terms of all of I's plays from rounds 0 and 1

Instead, we use the game tree to define covers $\{V_k^n\}_k$ as follows.
Let $V_k^0 = U_k$; then $\{V_k^0\}_k$ is increasing open cover
Next let

$$V_k^1 = U_k \cap U_{0k} \cap U_{1k} \cap \ldots \cap U_{k-1,k}.$$

Since we have assumed $U_{ik} \supset U_i$, note that $V_k^1 = \bigcap_{\sigma \in \omega^{\leq 1}} U_{\sigma k}$, i.e.,
$V_k^1 =$ intersection of all k^{th} terms of all of I's plays from rounds 0 and 1

Claim. $\{V_k^1\}_k$ *is an increasing open cover.*

Instead, we use the game tree to define covers $\{V_k^n\}_k$ as follows.
Let $V_k^0 = U_k$; then $\{V_k^0\}_k$ is increasing open cover
Next let

$$V_k^1 = U_k \cap U_{0k} \cap U_{1k} \cap \ldots \cap U_{k-1,k}.$$

Since we have assumed $U_{ik} \supset U_i$, note that $V_k^1 = \bigcap_{\sigma \in \omega^{\leq 1}} U_{\sigma k}$, i.e.,
$V_k^1 =$ intersection of all k^{th} terms of all of I's plays from rounds 0 and 1

Claim. $\{V_k^1\}_k$ *is an increasing open cover.*
Increasing: $V_{k+1}^1 = U_{k+1} \cap \bigcap_{i \leq k} U_{i,k+1}$

Instead, we use the game tree to define covers $\{V_k^n\}_k$ as follows.
Let $V_k^0 = U_k$; then $\{V_k^0\}_k$ is increasing open cover
Next let

$$V_k^1 = U_k \cap U_{0k} \cap U_{1k} \cap \ldots \cap U_{k-1,k}.$$

Since we have assumed $U_{ik} \supset U_i$, note that $V_k^1 = \bigcap_{\sigma \in \omega^{\leq 1}} U_{\sigma k}$, i.e.,
$V_k^1 =$ intersection of all k^{th} terms of all of I's plays from rounds 0 and 1

Claim. $\{V_k^1\}_k$ *is an increasing open cover.*
Increasing: $V_{k+1}^1 = U_{k+1} \cap \bigcap_{i \leq k} U_{i,k+1} \supseteq U_k \cap (\bigcap_{i<k} U_{ik}) \cap U_{k,k+1} =$
$U_k \cap (\bigcap_{i<k} U_{ik}) = V_k^1$.

Instead, we use the game tree to define covers $\{V_k^n\}_k$ as follows.
Let $V_k^0 = U_k$; then $\{V_k^0\}_k$ is increasing open cover
Next let

$$V_k^1 = U_k \cap U_{0k} \cap U_{1k} \cap \ldots \cap U_{k-1,k}.$$

Since we have assumed $U_{ik} \supset U_i$, note that $V_k^1 = \bigcap_{\sigma \in \omega^{\leq 1}} U_{\sigma k}$, i.e.,
$V_k^1 =$ intersection of all k^{th} terms of all of I's plays from rounds 0 and 1

Claim. $\{V_k^1\}_k$ *is an increasing open cover.*
Increasing: $V_{k+1}^1 = U_{k+1} \cap \bigcap_{i \leq k} U_{i,k+1} \supseteq U_k \cap (\bigcap_{i<k} U_{ik}) \cap U_{k,k+1} = U_k \cap (\bigcap_{i<k} U_{ik}) = V_k^1$.

Cover: Let $x \in X$. There is k_0 with $x \in U_{k_0}$.

Instead, we use the game tree to define covers $\{V_k^n\}_k$ as follows.
Let $V_k^0 = U_k$; then $\{V_k^0\}_k$ is increasing open cover
Next let

$$V_k^1 = U_k \cap U_{0k} \cap U_{1k} \cap \ldots \cap U_{k-1,k}.$$

Since we have assumed $U_{ik} \supset U_i$, note that $V_k^1 = \bigcap_{\sigma \in \omega^{\leq 1}} U_{\sigma k}$, i.e.,
$V_k^1 =$ intersection of all k^{th} terms of all of I's plays from rounds 0 and 1

Claim. $\{V_k^1\}_k$ *is an increasing open cover.*
Increasing: $V_{k+1}^1 = U_{k+1} \cap \bigcap_{i \leq k} U_{i,k+1} \supseteq U_k \cap (\bigcap_{i<k} U_{ik}) \cap U_{k,k+1} = U_k \cap (\bigcap_{i<k} U_{ik}) = V_k^1$.

Cover: Let $x \in X$. There is k_0 with $x \in U_{k_0}$.
For each $i < k_0$, there is l_i such that $x \in U_{il_i}$.

Instead, we use the game tree to define covers $\{V_k^n\}_k$ as follows.
Let $V_k^0 = U_k$; then $\{V_k^0\}_k$ is increasing open cover
Next let

$$V_k^1 = U_k \cap U_{0k} \cap U_{1k} \cap \ldots \cap U_{k-1,k}.$$

Since we have assumed $U_{ik} \supset U_i$, note that $V_k^1 = \bigcap_{\sigma \in \omega^{\leq 1}} U_{\sigma k}$, i.e.,
$V_k^1 =$ intersection of all k^{th} terms of all of I's plays from rounds 0 and 1

Claim. $\{V_k^1\}_k$ *is an increasing open cover.*
Increasing: $V_{k+1}^1 = U_{k+1} \cap \bigcap_{i \leq k} U_{i,k+1} \supseteq U_k \cap (\bigcap_{i<k} U_{ik}) \cap U_{k,k+1} = U_k \cap (\bigcap_{i<k} U_{ik}) = V_k^1$.

Cover: Let $x \in X$. There is k_0 with $x \in U_{k_0}$.
For each $i < k_0$, there is l_i such that $x \in U_{il_i}$.
Let k be greater than k_0 and the l_i's.

Instead, we use the game tree to define covers $\{V_k^n\}_k$ as follows.
Let $V_k^0 = U_k$; then $\{V_k^0\}_k$ is increasing open cover
Next let

$$V_k^1 = U_k \cap U_{0k} \cap U_{1k} \cap \ldots \cap U_{k-1,k}.$$

Since we have assumed $U_{ik} \supset U_i$, note that $V_k^1 = \bigcap_{\sigma \in \omega^{\leq 1}} U_{\sigma k}$, i.e.,
$V_k^1 =$ intersection of all k^{th} terms of all of I's plays from rounds 0 and 1

Claim. $\{V_k^1\}_k$ *is an increasing open cover.*
Increasing: $V_{k+1}^1 = U_{k+1} \cap \bigcap_{i \leq k} U_{i,k+1} \supseteq U_k \cap (\bigcap_{i<k} U_{ik}) \cap U_{k,k+1} =$
$U_k \cap (\bigcap_{i<k} U_{ik}) = V_k^1$.

Cover: Let $x \in X$. There is k_0 with $x \in U_{k_0}$.
For each $i < k_0$, there is l_i such that $x \in U_{il_i}$.
Let k be greater than k_0 and the l_i's.
Then V_k^1 is the intersection of (i)U_k;

Instead, we use the game tree to define covers $\{V_k^n\}_k$ as follows.
Let $V_k^0 = U_k$; then $\{V_k^0\}_k$ is increasing open cover
Next let

$$V_k^1 = U_k \cap U_{0k} \cap U_{1k} \cap \ldots \cap U_{k-1,k}.$$

Since we have assumed $U_{ik} \supset U_i$, note that $V_k^1 = \bigcap_{\sigma \in \omega^{\leq 1}} U_{\sigma k}$, i.e.,
$V_k^1 =$ intersection of all k^{th} terms of all of I's plays from rounds 0 and 1

Claim. $\{V_k^1\}_k$ *is an increasing open cover.*
Increasing: $V_{k+1}^1 = U_{k+1} \cap \bigcap_{i \leq k} U_{i,k+1} \supseteq U_k \cap (\bigcap_{i<k} U_{ik}) \cap U_{k,k+1} =$
$U_k \cap (\bigcap_{i<k} U_{ik}) = V_k^1$.

Cover: Let $x \in X$. There is k_0 with $x \in U_{k_0}$.
For each $i < k_0$, there is l_i such that $x \in U_{il_i}$.
Let k be greater than k_0 and the l_i's.
Then V_k^1 is the intersection of (i)U_k; (ii) $\bigcap_{i<k_0} U_{ik}$,

Instead, we use the game tree to define covers $\{V_k^n\}_k$ as follows.
Let $V_k^0 = U_k$; then $\{V_k^0\}_k$ is increasing open cover
Next let

$$V_k^1 = U_k \cap U_{0k} \cap U_{1k} \cap \ldots \cap U_{k-1,k}.$$

Since we have assumed $U_{ik} \supset U_i$, note that $V_k^1 = \bigcap_{\sigma \in \omega^{\leq 1}} U_{\sigma k}$, i.e.,
$V_k^1 =$ intersection of all k^{th} terms of all of I's plays from rounds 0 and 1

Claim. $\{V_k^1\}_k$ *is an increasing open cover.*
Increasing: $V_{k+1}^1 = U_{k+1} \cap \bigcap_{i \leq k} U_{i,k+1} \supseteq U_k \cap (\bigcap_{i<k} U_{ik}) \cap U_{k,k+1} = U_k \cap (\bigcap_{i<k} U_{ik}) = V_k^1$.

Cover: Let $x \in X$. There is k_0 with $x \in U_{k_0}$.
For each $i < k_0$, there is l_i such that $x \in U_{il_i}$.
Let k be greater than k_0 and the l_i's.
Then V_k^1 is the intersection of (i)U_k; (ii) $\bigcap_{i<k_0} U_{ik}$, and (iii) $\bigcap_{k_0 \leq i < k} U_{ik}$.

Instead, we use the game tree to define covers $\{V_k^n\}_k$ as follows.
Let $V_k^0 = U_k$; then $\{V_k^0\}_k$ is increasing open cover
Next let

$$V_k^1 = U_k \cap U_{0k} \cap U_{1k} \cap \ldots \cap U_{k-1,k}.$$

Since we have assumed $U_{ik} \supset U_i$, note that $V_k^1 = \bigcap_{\sigma \in \omega^{\leq 1}} U_{\sigma k}$, i.e.,
$V_k^1 =$ intersection of all k^{th} terms of all of I's plays from rounds 0 and 1

Claim. $\{V_k^1\}_k$ *is an increasing open cover.*
Increasing: $V_{k+1}^1 = U_{k+1} \cap \bigcap_{i \leq k} U_{i,k+1} \supseteq U_k \cap (\bigcap_{i<k} U_{ik}) \cap U_{k,k+1} =$
$U_k \cap (\bigcap_{i<k} U_{ik}) = V_k^1$.

Cover: Let $x \in X$. There is k_0 with $x \in U_{k_0}$.
For each $i < k_0$, there is l_i such that $x \in U_{il_i}$.
Let k be greater than k_0 and the l_i's.
Then V_k^1 is the intersection of (i)U_k; (ii) $\bigcap_{i<k_0} U_{ik}$, and (iii) $\bigcap_{k_0 \leq i < k} U_{ik}$.
Now x is in (i) since $k \geq k_0$,

Instead, we use the game tree to define covers $\{V_k^n\}_k$ as follows.
Let $V_k^0 = U_k$; then $\{V_k^0\}_k$ is increasing open cover
Next let

$$V_k^1 = U_k \cap U_{0k} \cap U_{1k} \cap \ldots \cap U_{k-1,k}.$$

Since we have assumed $U_{ik} \supset U_i$, note that $V_k^1 = \bigcap_{\sigma \in \omega^{\leq 1}} U_{\sigma k}$, i.e.,
V_k^1 = intersection of all k^{th} terms of all of I's plays from rounds 0 and 1

Claim. $\{V_k^1\}_k$ *is an increasing open cover.*
Increasing: $V_{k+1}^1 = U_{k+1} \cap \bigcap_{i \leq k} U_{i,k+1} \supseteq U_k \cap (\bigcap_{i<k} U_{ik}) \cap U_{k,k+1} = U_k \cap (\bigcap_{i<k} U_{ik}) = V_k^1$.

Cover: Let $x \in X$. There is k_0 with $x \in U_{k_0}$.
For each $i < k_0$, there is l_i such that $x \in U_{il_i}$.
Let k be greater than k_0 and the l_i's.
Then V_k^1 is the intersection of (i)U_k; (ii) $\bigcap_{i<k_0} U_{ik}$, and (iii) $\bigcap_{k_0 \leq i < k} U_{ik}$.
Now x is in (i) since $k \geq k_0$,
x is in (ii) since $k \geq l_i$ for all $i < k_0$, and

Instead, we use the game tree to define covers $\{V_k^n\}_k$ as follows.
Let $V_k^0 = U_k$; then $\{V_k^0\}_k$ is increasing open cover
Next let

$$V_k^1 = U_k \cap U_{0k} \cap U_{1k} \cap \ldots \cap U_{k-1,k}.$$

Since we have assumed $U_{ik} \supset U_i$, note that $V_k^1 = \bigcap_{\sigma \in \omega^{\leq 1}} U_{\sigma k}$, i.e.,
$V_k^1 =$ intersection of all k^{th} terms of all of I's plays from rounds 0 and 1

Claim. $\{V_k^1\}_k$ *is an increasing open cover.*
Increasing: $V_{k+1}^1 = U_{k+1} \cap \bigcap_{i \leq k} U_{i,k+1} \supseteq U_k \cap (\bigcap_{i<k} U_{ik}) \cap U_{k,k+1} = U_k \cap (\bigcap_{i<k} U_{ik}) = V_k^1$.

Cover: Let $x \in X$. There is k_0 with $x \in U_{k_0}$.
For each $i < k_0$, there is l_i such that $x \in U_{il_i}$.
Let k be greater than k_0 and the l_i's.
Then V_k^1 is the intersection of (i)U_k; (ii) $\bigcap_{i<k_0} U_{ik}$, and (iii) $\bigcap_{k_0 \leq i<k} U_{ik}$.
Now x is in (i) since $k \geq k_0$,
x is in (ii) since $k \geq l_i$ for all $i < k_0$, and
x is in (iii) since $U_{ik} \supseteq U_{k_0}$ for all $i \geq k_0$.

Instead, we use the game tree to define covers $\{V_k^n\}_k$ as follows.
Let $V_k^0 = U_k$; then $\{V_k^0\}_k$ is increasing open cover
Next let

$$V_k^1 = U_k \cap U_{0k} \cap U_{1k} \cap \ldots \cap U_{k-1,k}.$$

Since we have assumed $U_{ik} \supset U_i$, note that $V_k^1 = \bigcap_{\sigma \in \omega^{\leq 1}} U_{\sigma k}$, i.e.,
$V_k^1 =$ intersection of all k^{th} terms of all of I's plays from rounds 0 and 1

Claim. $\{V_k^1\}_k$ *is an increasing open cover.*
Increasing: $V_{k+1}^1 = U_{k+1} \cap \bigcap_{i \leq k} U_{i,k+1} \supseteq U_k \cap (\bigcap_{i<k} U_{ik}) \cap U_{k,k+1} = U_k \cap (\bigcap_{i<k} U_{ik}) = V_k^1$.

Cover: Let $x \in X$. There is k_0 with $x \in U_{k_0}$.
For each $i < k_0$, there is l_i such that $x \in U_{il_i}$.
Let k be greater than k_0 and the l_i's.
Then V_k^1 is the intersection of (i)U_k; (ii) $\bigcap_{i<k_0} U_{ik}$, and (iii) $\bigcap_{k_0 \leq i < k} U_{ik}$.
Now x is in (i) since $k \geq k_0$,
x is in (ii) since $k \geq l_i$ for all $i < k_0$, and
x is in (iii) since $U_{ik} \supseteq U_{k_0}$ for all $i \geq k_0$.
So $x \in V_k^1$.

Similarly, for each n, $\{V_k^n\}_k$ is an increasing open cover, where

$$V_k^n = \bigcap_{\sigma \in \omega^{\leq n}} U_{\sigma^\frown k}$$

Similarly, for each n, $\{V_k^n\}_k$ is an increasing open cover, where

$$V_k^n = \bigcap_{\sigma \in \omega^{\leq n}} U_{\sigma^\frown k}$$

Since X is Menger, there is $f : \omega \to \omega$ such that $X = \bigcup_{n \in \omega} V_{f(n)}^n$.

Similarly, for each n, $\{V_k^n\}_k$ is an increasing open cover, where

$$V_k^n = \bigcap_{\sigma \in \omega^{\leq n}} U_{\sigma^\frown k}$$

Since X is Menger, there is $f : \omega \to \omega$ such that $X = \bigcup_{n \in \omega} V_{f(n)}^n$.

But $V_{f(n)}^n \subset U_{(f \restriction n)^\frown f(n)} = U_{f \restriction n+1}$.

Similarly, for each n, $\{V_k^n\}_k$ is an increasing open cover, where

$$V_k^n = \bigcap_{\sigma \in \omega^{\leq n}} U_{\sigma^\frown k}$$

Since X is Menger, there is $f : \omega \to \omega$ such that $X = \bigcup_{n \in \omega} V_{f(n)}^n$.

But $V_{f(n)}^n \subset U_{(f \restriction n)^\frown f(n)} = U_{f \restriction n+1}$.

So $X = \bigcup_{n \in \omega} U_{f \restriction n+1}$, which corresponds to a play of the game in which I's strategy has been defeated. $\square$

TUTORIAL: Topological games, lecture II

Convergence game

Let X be a space, and H a closed subset of X. Define $G(H, X)$ as follows:

Convergence game

Let X be a space, and H a closed subset of X. Define $G(H, X)$ as follows:

There are two players, O and P.

Convergence game

Let X be a space, and H a closed subset of X. Define $G(H,X)$ as follows:

There are two players, O and P.
In the nth round, O chooses an open $O_n \supset H$, and P chooses a point $p_n \in O_n$.

Convergence game

Let X be a space, and H a closed subset of X. Define $G(H, X)$ as follows:

There are two players, O and P.
In the nth round, O chooses an open $O_n \supset H$, and P chooses a point $p_n \in O_n$.

We say O *wins* the game if $p_n \rightarrow H$ in the sense that every open superset of H contains p_n for all but finitely many $n \in \omega$.

Convergence game

Let X be a space, and H a closed subset of X. Define $G(H, X)$ as follows:

There are two players, O and P.
In the nth round, O chooses an open $O_n \supset H$, and P chooses a point $p_n \in O_n$.

We say O *wins* the game if $p_n \to H$ in the sense that every open superset of H contains p_n for all but finitely many $n \in \omega$.

If O $\uparrow G(H, X)$, we call H a *W-set* in X.

Clearly X first countable implies every point of X is a W-set in X.

Clearly X first countable implies every point of X is a W-set in X. Spaces in which O has a winning strategy at every point are called W-spaces.

Clearly X first countable implies every point of X is a W-set in X. Spaces in which O has a winning strategy at every point are called W-spaces.

A W-space which is not first countable:

Clearly X first countable implies every point of X is a W-set in X. Spaces in which O has a winning strategy at every point are called W-spaces.

A W-space which is not first countable:
The one-point compactification of uncountable discrete space

Clearly X first countable implies every point of X is a W-set in X. Spaces in which O has a winning strategy at every point are called W-spaces.

A W-space which is not first countable:
The one-point compactification of uncountable discrete space

I called spaces in which P does not have a winning strategy w-spaces.

Clearly X first countable implies every point of X is a W-set in X. Spaces in which O has a winning strategy at every point are called W-spaces.

A W-space which is not first countable:
The one-point compactification of uncountable discrete space

I called spaces in which P does not have a winning strategy w-spaces.

P. Sharma proved X is a w-space iff for each $x \in X$:
if $x \in \overline{A}_n$ for each $n \in \omega$, then there are $x_n \in A_n$ with $x_n \to x$.

Clearly X first countable implies every point of X is a W-set in X. Spaces in which O has a winning strategy at every point are called W-spaces.

A W-space which is not first countable:
The one-point compactification of uncountable discrete space

I called spaces in which P does not have a winning strategy w-spaces.

P. Sharma proved X is a w-space iff for each $x \in X$:
if $x \in \overline{A}_n$ for each $n \in \omega$, then there are $x_n \in A_n$ with $x_n \to x$.

This showed the class of w-spaces equivalent to a class introduced by Arhangel'skii (Fréchet α_2-spaces).

Theorem

1. *First-countable* $\Rightarrow$ *W-space* $\Rightarrow$ *w-space* $\Rightarrow$ *Fréchet;*
2. *W-spaces are hereditary and countably productive (closed under Σ-products, even).*

Theorem

1. *First-countable* $\Rightarrow$ W*-space* $\Rightarrow$ w*-space* $\Rightarrow$ *Fréchet;*
2. W*-spaces are hereditary and countably productive (closed under* Σ*-products, even).*

Consider the game $G(H, X)$ played in a compact Hausdorff space X, where $H \subset X$ is closed.

Theorem

1. *First-countable* $\Rightarrow$ *W-space* $\Rightarrow$ *w-space* $\Rightarrow$ *Fréchet;*
2. *W-spaces are hereditary and countably productive (closed under Σ-products, even).*

Consider the game $G(H, X)$ played in a compact Hausdorff space X, where $H \subset X$ is closed.

O will have a winning strategy if H has a countable "outer base", i.e., there is a countable collection of open supersets of H such that every open superset of H contains one. By compactness, H has a countable outer base iff H is G_δ.

Theorem

1. *First-countable $\Rightarrow$ W-space $\Rightarrow$ w-space $\Rightarrow$ Fréchet;*
2. *W-spaces are hereditary and countably productive (closed under Σ-products, even).*

Consider the game $G(H, X)$ played in a compact Hausdorff space X, where $H \subset X$ is closed.

O will have a winning strategy if H has a countable "outer base", i.e., there is a countable collection of open supersets of H such that every open superset of H contains one. By compactness, H has a countable outer base iff H is G_δ.

A classical result:

Theorem (Schneider)

A compact Hausdorff space X is metrizable iff the diagonal Δ of X is G_δ in X^2.

O can have a winning strategy in $G(H, X)$ for compact X without H being G_δ:

O can have a winning strategy in $G(H, X)$ for compact X without H being G_δ:
consider one-point compactification of uncountable discrete space.

O can have a winning strategy in $G(H, X)$ for compact X without H being G_δ:
consider one-point compactification of uncountable discrete space.

To say that O has winning strategy in $G(\Delta, X^2)$ is weaker than to say the diagonal is G_δ in X^2.

O can have a winning strategy in $G(H, X)$ for compact X without H being G_δ:
consider one-point compactification of uncountable discrete space.

To say that O has winning strategy in $G(\Delta, X^2)$ is weaker than to say the diagonal is G_δ in X^2.

Definition. Compact X is Corson compact iff

O can have a winning strategy in $G(H, X)$ for compact X without H being G_δ:
consider one-point compactification of uncountable discrete space.

To say that O has winning strategy in $G(\Delta, X^2)$ is weaker than to say the diagonal is G_δ in X^2.

Definition. Compact X is Corson compact iff

- X is homeo to subspace of

$$\Sigma\mathbb{R}^\kappa = \{x \in \mathbb{R}^\kappa : |\{\alpha < \kappa : x(\alpha) \neq 0\}| \leq \omega\}$$

  or equivalently

O can have a winning strategy in $G(H, X)$ for compact X without H being G_δ:
consider one-point compactification of uncountable discrete space.

To say that O has winning strategy in $G(\Delta, X^2)$ is weaker than to say the diagonal is G_δ in X^2.

Definition. Compact X is Corson compact iff

- X is homeo to subspace of

$$\Sigma\mathbb{R}^\kappa = \{x \in \mathbb{R}^\kappa : |\{\alpha < \kappa : x(\alpha) \neq 0\}| \leq \omega\}$$

  or equivalently

- X has a point-countable T_0-separating cover by open F_σ's.

O can have a winning strategy in $G(H, X)$ for compact X without H being G_δ:
consider one-point compactification of uncountable discrete space.

To say that O has winning strategy in $G(\Delta, X^2)$ is weaker than to say the diagonal is G_δ in X^2.

Definition. Compact X is Corson compact iff

- X is homeo to subspace of

$$\Sigma\mathbb{R}^\kappa = \{x \in \mathbb{R}^\kappa : |\{\alpha < \kappa : x(\alpha) \neq 0\}| \leq \omega\}$$

  or equivalently
- X has a point-countable T_0-separating cover by open F_σ's.

Theorem(G.G., 1984)

A compact space X is Corson compact iff O $\uparrow G(\Delta, X^2)$.

Eberlein compact: homeo to weakly compact subset of Banach space

Eberlein compact: homeo to weakly compact subset of Banach space

Eberlein $\Rightarrow$ Corson

Eberlein compact: homeo to weakly compact subset of Banach space

Eberlein $\Rightarrow$ Corson

Theorem(G.G., 1986)

A compact X is Eberlein compact iff O has a winning strategy in $G(\Delta, X^2)$ which depends only on P's last move and the number of the move (i.e., a Markov strategy).

Eberlein compact: homeo to weakly compact subset of Banach space

Eberlein $\Rightarrow$ Corson

Theorem(G.G., 1986)

A compact X is Eberlein compact iff O has a winning strategy in $G(\Delta, X^2)$ which depends only on P's last move and the number of the move (i.e., a Markov strategy).

Strong Eberlein = scattered Eberlein

Eberlein compact: homeo to weakly compact subset of Banach space

Eberlein $\Rightarrow$ Corson

Theorem(G.G., 1986)

A compact X is Eberlein compact iff O has a winning strategy in $G(\Delta, X^2)$ which depends only on P's last move and the number of the move (i.e., a Markov strategy).

Strong Eberlein = scattered Eberlein

Theorem(G.G., 1984)

A compact scattered space X is strong Eberlein compact iff X is a W-space.

A recent use of the Corson result.

A recent use of the Corson result.

DEF (Tkachuk): A space X is monotonically monolithic if one can assign to each $F \in [X]^{<\omega}$ a countable collection $\mathcal{N}(F)$ of subsets of X such that

A recent use of the Corson result.

DEF (Tkachuk): A space X is monotonically monolithic if one can assign to each $F \in [X]^{<\omega}$ a countable collection $\mathcal{N}(F)$ of subsets of X such that

1. $F \subset F' \Rightarrow \mathcal{N}(F) \subset \mathcal{N}(F')$;

A recent use of the Corson result.

DEF (Tkachuk): A space X is monotonically monolithic if one can assign to each $F \in [X]^{<\omega}$ a countable collection $\mathcal{N}(F)$ of subsets of X such that

1. $F \subset F' \Rightarrow \mathcal{N}(F) \subset \mathcal{N}(F')$;
2. If U open and $x \in \overline{A} \cap U$ there is $F \in [A]^{<\omega}$ and $N \in \mathcal{N}(F)$ with $x \in N \subset U$.

A recent use of the Corson result.

DEF (Tkachuk): A space X is monotonically monolithic if one can assign to each $F \in [X]^{<\omega}$ a countable collection $\mathcal{N}(F)$ of subsets of X such that

1. $F \subset F' \Rightarrow \mathcal{N}(F) \subset \mathcal{N}(F')$;
2. If U open and $x \in \overline{A} \cap U$ there is $F \in [A]^{<\omega}$ and $N \in \mathcal{N}(F)$ with $x \in N \subset U$.

(That is, $\bigcup_{F \in [A]^{<\omega}} \mathcal{N}(F)$ includes a network at every point of $\overline{A}$.)

Tkachuk proved:

1. Point-countable base $\Rightarrow$ monotonically monolithic

Tkachuk proved:

1. Point-countable base $\Rightarrow$ monotonically monolithic
2. X Lindelöf $\Sigma \Rightarrow C_p(X)$ monotonically monolithic

Tkachuk proved:

1. Point-countable base $\Rightarrow$ monotonically monolithic
2. X Lindelöf $\Sigma \Rightarrow C_p(X)$ monotonically monolithic (Lindelöf $\Sigma =$ continuous image of closed subspace of separable metric $\times$ compact)

Tkachuk proved:

1. Point-countable base $\Rightarrow$ monotonically monolithic
2. X Lindelöf $\Sigma \Rightarrow C_p(X)$ monotonically monolithic (Lindelöf Σ = continuous image of closed subspace of separable metric $\times$ compact)
3. Monotonically monolithic $\Rightarrow$ hereditarily D.

D-space: $x \in N(x)^o\ \forall x \in X \Rightarrow \exists$ closed discrete D with $\{N(x) : x \in D\}$ covering X

Tkachuk proved:

1. Point-countable base $\Rightarrow$ monotonically monolithic
2. X Lindelöf $\Sigma \Rightarrow C_p(X)$ monotonically monolithic (Lindelöf $\Sigma =$ continuous image of closed subspace of separable metric $\times$ compact)
3. Monotonically monolithic $\Rightarrow$ hereditarily D.

D-space: $x \in N(x)^o \ \forall x \in X \Rightarrow \exists$ closed discrete D with $\{N(x) : x \in D\}$ covering X

X is Gul'ko compact iff X compact and $C_p(X)$ is Lindelöf Σ. It follows that:

Tkachuk proved:

1. Point-countable base $\Rightarrow$ monotonically monolithic
2. X Lindelöf Σ $\Rightarrow$ $C_p(X)$ monotonically monolithic (Lindelöf Σ = continuous image of closed subspace of separable metric $\times$ compact)
3. Monotonically monolithic $\Rightarrow$ hereditarily D.

D-space: $x \in N(x)^o \ \forall x \in X \Rightarrow \exists$ closed discrete D with $\{N(x) : x \in D\}$ covering X

X is Gul'ko compact iff X compact and $C_p(X)$ is Lindelöf Σ. It follows that:

$$\text{Gul'ko compact} \Rightarrow \text{monotonically monolithic}$$

Tkachuk proved:

1. Point-countable base $\Rightarrow$ monotonically monolithic
2. X Lindelöf $\Sigma \Rightarrow C_p(X)$ monotonically monolithic (Lindelöf $\Sigma =$ continuous image of closed subspace of separable metric $\times$ compact)
3. Monotonically monolithic $\Rightarrow$ hereditarily D.

D-space: $x \in N(x)^o \; \forall x \in X \Rightarrow \exists$ closed discrete D with $\{N(x) : x \in D\}$ covering X

X is Gul'ko compact iff X compact and $C_p(X)$ is Lindelöf Σ. It follows that:

Gul'ko compact $\Rightarrow$ monotonically monolithic

Eberlein compact $\Rightarrow$ Gul'ko compact $\Rightarrow$ Corson compact

Tkachuk proved:

1. Point-countable base $\Rightarrow$ monotonically monolithic
2. X Lindelöf $\Sigma \Rightarrow C_p(X)$ monotonically monolithic (Lindelöf $\Sigma =$ continuous image of closed subspace of separable metric $\times$ compact)
3. Monotonically monolithic $\Rightarrow$ hereditarily D.

D-space: $x \in N(x)^o \ \forall x \in X \Rightarrow \exists$ closed discrete D with $\{N(x) : x \in D\}$ covering X

X is Gul'ko compact iff X compact and $C_p(X)$ is Lindelöf Σ. It follows that:

$$\text{Gul'ko compact} \Rightarrow \text{monotonically monolithic}$$

Eberlein compact $\Rightarrow$ Gul'ko compact $\Rightarrow$ Corson compact

$\Rightarrow$ hereditarily D

Question(Tkachuk)

- Monotonically monolithic compact $\Rightarrow$ Corson compact?

Question(Tkachuk)

- Monotonically monolithic compact $\Rightarrow$ Corson compact?

Yes:

Theorem (G.G., 2010)

If X is compact and monotonically monolithic, then X is Corson compact.

Lemma

Suppose X is compact and monotonically monolithic. Then O has a winning strategy in $G(H, X)$ for any closed $H \subset X$.

Lemma

Suppose X is compact and monotonically monolithic. Then O has a winning strategy in $G(H, X)$ for any closed $H \subset X$.

To prove the Theorem from the Lemma:

Lemma

Suppose X is compact and monotonically monolithic. Then O has a winning strategy in $G(H, X)$ for any closed $H \subset X$.

To prove the Theorem from the Lemma:

X compact monotonically monolithic $\Rightarrow$ ditto for X^2 $\Rightarrow$ O has winning strategy in $G(\Delta, X^2)$ $\Rightarrow$ X Corson compact

Proof of lemma

Proof of lemma

X compact, $\mathcal{N}$ witnesses monotonically monolithic.

Proof of lemma

X compact, $\mathcal{N}$ witnesses monotonically monolithic.

$H \subset X$ closed

Proof of lemma

X compact, $\mathcal{N}$ witnesses monotonically monolithic.

$H \subset X$ closed

O plays $O_0 = X$, P plays $p_0 \in X$.

Proof of lemma

X compact, $\mathcal{N}$ witnesses monotonically monolithic.

$H \subset X$ closed

O plays $O_0 = X$, P plays $p_0 \in X$.

O looks at $\mathcal{N}(\{p_0\}) = \{N_{00}, N_{01}, \dots\}$

Proof of lemma

X compact, $\mathcal{N}$ witnesses monotonically monolithic.

$H \subset X$ closed

O plays $O_0 = X$, P plays $p_0 \in X$.

O looks at $\mathcal{N}(\{p_0\}) = \{N_{00}, N_{01}, \dots\}$

O chooses open $O_1 \supset H$ s.t. $\overline{O}_1 \cap N_{00} = \emptyset$ if such O_1 exists; else $O_1 = X$.

Proof of lemma

X compact, $\mathcal{N}$ witnesses monotonically monolithic.

$H \subset X$ closed

O plays $O_0 = X$, P plays $p_0 \in X$.

O looks at $\mathcal{N}(\{p_0\}) = \{N_{00}, N_{01}, \dots\}$

O chooses open $O_1 \supset H$ s.t. $\overline{O}_1 \cap N_{00} = \emptyset$ if such O_1 exists; else $O_1 = X$.

P chooses $p_1 \in O_1$. Let $\mathcal{N}(\{p_0, p_1\}) = \{N_{10}, N_{11}, \dots\}$

Proof of lemma

X compact, $\mathcal{N}$ witnesses monotonically monolithic.

$H \subset X$ closed

O plays $O_0 = X$, P plays $p_0 \in X$.

O looks at $\mathcal{N}(\{p_0\}) = \{N_{00}, N_{01}, \dots\}$

O chooses open $O_1 \supset H$ s.t. $\overline{O}_1 \cap N_{00} = \emptyset$ if such O_1 exists; else $O_1 = X$.

P chooses $p_1 \in O_1$. Let $\mathcal{N}(\{p_0, p_1\}) = \{N_{10}, N_{11}, \dots\}$

O chooses $O_2 \subset O_1$ s.t., whenever possible for $i, j < 2$, $\overline{O}_2 \cap N_{ij} = \emptyset$.

Proof of lemma

X compact, $\mathcal{N}$ witnesses monotonically monolithic.

$H \subset X$ closed

O plays $O_0 = X$, P plays $p_0 \in X$.

O looks at $\mathcal{N}(\{p_0\}) = \{N_{00}, N_{01}, \ldots\}$

O chooses open $O_1 \supset H$ s.t. $\overline{O}_1 \cap N_{00} = \emptyset$ if such O_1 exists; else $O_1 = X$.

P chooses $p_1 \in O_1$. Let $\mathcal{N}(\{p_0, p_1\}) = \{N_{10}, N_{11}, \ldots\}$

O chooses $O_2 \subset O_1$ s.t., whenever possible for $i, j < 2$, $\overline{O}_2 \cap N_{ij} = \emptyset$.

Etc

Claim. This wins for O.

Claim. This wins for O.

Suppose $p_n \not\to H$. Then $\{p_n\}_n$ has limit point $q \notin H$

Claim. This wins for O.

Suppose $p_n \not\to H$. Then $\{p_n\}_n$ has limit point $q \notin H$

Let $q \in U$ open, $\overline{U} \cap H = \emptyset$.

Claim. This wins for O.

Suppose $p_n \not\to H$. Then $\{p_n\}_n$ has limit point $q \notin H$

Let $q \in U$ open, $\overline{U} \cap H = \emptyset$.

$\exists k \in \omega$ with $N \in \mathcal{N}(\{p_i\}_{i \leq k})$ and $q \in N \subset U$

Claim. This wins for O.

Suppose $p_n \not\to H$. Then $\{p_n\}_n$ has limit point $q \notin H$

Let $q \in U$ open, $\overline{U} \cap H = \emptyset$.

$\exists k \in \omega$ with $N \in \mathcal{N}(\{p_i\}_{i \leq k})$ and $q \in N \subset U$

$N = N_{kj}$ for some k, j

Claim. This wins for O.

Suppose $p_n \not\to H$. Then $\{p_n\}_n$ has limit point $q \notin H$

Let $q \in U$ open, $\overline{U} \cap H = \emptyset$.

$\exists k \in \omega$ with $N \in \mathcal{N}(\{p_i\}_{i \leq k})$ and $q \in N \subset U$

$N = N_{kj}$ for some k, j

$\overline{O}_n \cap N = \emptyset$ for $n > max\{j, k\}$

Claim. This wins for O.
Suppose $p_n \not\to H$. Then $\{p_n\}_n$ has limit point $q \notin H$

Let $q \in U$ open, $\overline{U} \cap H = \emptyset$.

$\exists k \in \omega$ with $N \in \mathcal{N}(\{p_i\}_{i \leq k})$ and $q \in N \subset U$

$N = N_{kj}$ for some k, j

$\overline{O}_n \cap N = \emptyset$ for $n > max\{j, k\}$

$q \notin \overline{O}_n \Rightarrow q$ not limit of $\{p_n\}_n$.
Contradiction.

How to show O wins in $G(\Delta, X^2)$ implies X Corson compact.

How to show O wins in $G(\Delta, X^2)$ implies X Corson compact.

First, easy to get from this that O wins in $G(x, X)$ for every $x \in X$; i.e., X is a W-space hence Fréchet (and so is X^2).

How to show O wins in $G(\Delta, X^2)$ implies X Corson compact.

First, easy to get from this that O wins in $G(x, X)$ for every $x \in X$; i.e., X is a W-space hence Fréchet (and so is X^2).

Use the point-countable T_0-separating open cover characterization. Can get such if can show $X^2 \setminus \Delta$ is metalindelöf, i.e., every open cover has a point-countable open refinement.

How to show O wins in $G(\Delta, X^2)$ implies X Corson compact.

First, easy to get from this that O wins in $G(x, X)$ for every $x \in X$; i.e., X is a W-space hence Fréchet (and so is X^2).

Use the point-countable T_0-separating open cover characterization. Can get such if can show $X^2 \setminus \Delta$ is metalindelöf, i.e., every open cover has a point-countable open refinement.
Then we have:

Theorem

Let X be compact and countably tight, and H closed. Then O has a winning strategy in $G(H, X)$ iff $X \setminus H$ is metalindelöf.

It is useful to view the game as a game in $X \setminus H$, with players K and P.

It is useful to view the game as a game in $X \setminus H$, with players K and P.

In the n^{th} round, K chooses a compact $K_n \subset X \setminus H$ (the complement of a play by O), and P responds with a point $p_n \notin K_n$.

It is useful to view the game as a game in $X \setminus H$, with players K and P.

In the n^{th} round, K chooses a compact $K_n \subset X \setminus H$ (the complement of a play by O), and P responds with a point $p_n \notin K_n$.

K *wins* if $p_n \to \infty$ (i.e., $\{p_n : n \in \omega\}$ is closed discrete in $X \setminus H$). Replacing $X \setminus H$ with X, let us denote this game by $G_{K,P}(X)$.

It is useful to view the game as a game in $X \setminus H$, with players K and P.

In the n^{th} round, K chooses a compact $K_n \subset X \setminus H$ (the complement of a play by O), and P responds with a point $p_n \notin K_n$.

K *wins* if $p_n \to \infty$ (i.e., $\{p_n : n \in \omega\}$ is closed discrete in $X \setminus H$). Replacing $X \setminus H$ with X, let us denote this game by $G_{K,P}(X)$.

Then the result becomes:

Theorem

Let X be locally compact and countably tight. Then K has a winning strategy in $G_{K,P}(X)$ iff X is metalindelöf.

(I don't know if the countable tightness assumption is necessary.)

Outline of proof, elementary submodel style

Outline of proof, elementary submodel style

Proof. If X is metalindelöf, then there is a point-countable cover $\mathcal{U}$ of X by open sets with compact closures .

Outline of proof, elementary submodel style

Proof. If X is metalindelöf, then there is a point-countable cover $\mathcal{U}$ of X by open sets with compact closures .

K wins by looking at the countably many members of $\mathcal{U}$ containing P's chosen point at each round, and choosing an increasing sequence of compact sets that eventually cover every one of these members of $\mathcal{U}$. It is easy to check that this wins for K.

Outline of proof, elementary submodel style

Proof. If X is metalindelöf, then there is a point-countable cover $\mathcal{U}$ of X by open sets with compact closures .

K wins by looking at the countably many members of $\mathcal{U}$ containing P's chosen point at each round, and choosing an increasing sequence of compact sets that eventually cover every one of these members of $\mathcal{U}$. It is easy to check that this wins for K.

Now suppose K has a winning strategy σ, and let $\mathcal{U}$ be a cover of X by open sets with compact closures. Let M be an elementary submodel (of some sufficiently large $H(\theta)$) with $X, \mathcal{U}, \sigma \in M$.

Key Claim. $\overline{M \cap X} \subset \bigcup (M \cap \mathcal{U})$.

Key Claim. $\overline{M \cap X} \subset \bigcup(M \cap \mathcal{U})$.

Proof of Key Claim. Suppose $p \in \overline{M \cap X} \setminus \bigcup(M \cap \mathcal{U})$. Let $p \in U_p \in \mathcal{U}$.

Key Claim. $\overline{M \cap X} \subset \bigcup(M \cap \mathcal{U})$.

Proof of Key Claim. Suppose $p \in \overline{M \cap X} \setminus \bigcup(M \cap \mathcal{U})$. Let $p \in U_p \in \mathcal{U}$.

Suppose $F = \{p_0, p_1, \ldots, p_n\} \subset U_p \cap (M \cap X)$. Then $\sigma(F)$ is compact and in M so there exists a finite $\mathcal{U}_0 \subset \mathcal{U}$ in M covering $\sigma(F)$.

Key Claim. $\overline{M \cap X} \subset \bigcup(M \cap \mathcal{U})$.

Proof of Key Claim. Suppose $p \in \overline{M \cap X} \setminus \bigcup(M \cap \mathcal{U})$. Let $p \in U_p \in \mathcal{U}$.

Suppose $F = \{p_0, p_1, \ldots, p_n\} \subset U_p \cap (M \cap X)$. Then $\sigma(F)$ is compact and in M so there exists a finite $\mathcal{U}_0 \subset \mathcal{U}$ in M covering $\sigma(F)$.

Since M also contains a finite subset of $\mathcal{U}$ covering $\overline{\cup \mathcal{U}_0}$, we have $p \notin \overline{\cup \mathcal{U}_0}$. So there exists $p_{n+1} \in U_p \cap (M \cap X) \setminus \overline{\cup \mathcal{U}_0}$.

Key Claim. $\overline{M \cap X} \subset \bigcup(M \cap \mathcal{U})$.

Proof of Key Claim. Suppose $p \in \overline{M \cap X} \setminus \bigcup(M \cap \mathcal{U})$. Let $p \in U_p \in \mathcal{U}$.

Suppose $F = \{p_0, p_1, \ldots, p_n\} \subset U_p \cap (M \cap X)$. Then $\sigma(F)$ is compact and in M so there exists a finite $\mathcal{U}_0 \subset \mathcal{U}$ in M covering $\sigma(F)$.

Since M also contains a finite subset of $\mathcal{U}$ covering $\overline{\cup \mathcal{U}_0}$, we have $p \notin \overline{\cup \mathcal{U}_0}$. So there exists $p_{n+1} \in U_p \cap (M \cap X) \setminus \overline{\cup \mathcal{U}_0}$.

It follows that if K uses the strategy σ, P can always choose a point in $U_p \cap (M \cap X)$. But then K loses the game, a contradiction which completes the proof of Key Claim.

Since there is an M with $\mathcal{U} \subset M$, the next claim completes the proof of the theorem.

Since there is an M with $\mathcal{U} \subset M$, the next claim completes the proof of the theorem.

Claim 2. There is a point-countable open refinement $\mathcal{V}_M$ of $M \cap \mathcal{U}$ covering $\cup(M \cap \mathcal{U})$.

Since there is an M with $\mathcal{U} \subset M$, the next claim completes the proof of the theorem.

Claim 2. There is a point-countable open refinement $\mathcal{V}_M$ of $M \cap \mathcal{U}$ covering $\cup(M \cap \mathcal{U})$.

Proof of Claim 2. By induction on $|M| = \kappa$. Write $M = \bigcup\{M_\alpha : \alpha < \kappa\}$ and use Key Claim to put together point-countable refinements of $M_\alpha \cap \mathcal{U}$.

$G_{K,L}(X)$ is defined just like $G_{K,P}(X)$, except that P, who will be renamed L, chooses compact sets instead of points, i.e., L's n^{th} play is a compact set L_n missing K's previous move K_n.

$G_{K,L}(X)$ is defined just like $G_{K,P}(X)$, except that P, who will be renamed L, chooses compact sets instead of points, i.e., L's n^{th} play is a compact set L_n missing K's previous move K_n.

K wins iff $\{L_i\}_{i\in\omega}$ is a discrete collection.

$G_{K,L}(X)$ is defined just like $G_{K,P}(X)$, except that P, who will be renamed L, chooses compact sets instead of points, i.e., L's n^{th} play is a compact set L_n missing K's previous move K_n.

K wins iff $\{L_i\}_{i\in\omega}$ is a discrete collection.

$G^o_{K,L}(X)$ is the same as $G_{K,L}(X)$, except that K wins iff $\{L_i\}_{i\in\omega}$ has a discrete open expansion.

$G_{K,L}(X)$ is defined just like $G_{K,P}(X)$, except that P, who will be renamed L, chooses compact sets instead of points, i.e., L's n^{th} play is a compact set L_n missing K's previous move K_n.

K wins iff $\{L_i\}_{i \in \omega}$ is a discrete collection.

$G^o_{K,L}(X)$ is the same as $G_{K,L}(X)$, except that K wins iff $\{L_i\}_{i \in \omega}$ has a discrete open expansion.

It is easy to see that K has a winning strategy in any locally compact σ-compact space: K simply chooses at the n^{th} play the n^{th} set in an increasing sequence of compact sets whose interiors cover the space.

$G_{K,L}(X)$ is defined just like $G_{K,P}(X)$, except that P, who will be renamed L, chooses compact sets instead of points, i.e., L's n^{th} play is a compact set L_n missing K's previous move K_n.

K wins iff $\{L_i\}_{i\in\omega}$ is a discrete collection.

$G^o_{K,L}(X)$ is the same as $G_{K,L}(X)$, except that K wins iff $\{L_i\}_{i\in\omega}$ has a discrete open expansion.

It is easy to see that K has a winning strategy in any locally compact σ-compact space: K simply chooses at the n^{th} play the n^{th} set in an increasing sequence of compact sets whose interiors cover the space.

It is nearly as easy to see that K wins if X is a topological sum of locally compact σ-compact spaces, i.e., whenever X is locally compact and paracompact. The next theorem shows we have an equivalence:

Theorem

Let X be a locally compact space. Then the following are equivalent:

1. $K \uparrow G_{K,L}(X)$;
2. $K \uparrow G^{o}_{K,L}(X)$;
3. X *is paracompact.*

Why $G^{o}_{K,L}(X)$? Because it is the most natural one for attacking the following open problem:

Why $G^{o}_{K,L}(X)$? Because it is the most natural one for attacking the following open problem:

Question

For what (completely regular) spaces X is $C_k(X)$ a Baire space?

($C_k(X)$ is the space of continuous real-valued functions on X with the compact-open topology.)

Theorem (McCoy, Ntantu)

1. If NE $\uparrow$ BM($C_k(X)$) then K $\uparrow$ $G^o_{K,L}(X)$;
2. If $C_k(X)$ is Baire, then L $\not\uparrow$ $G^o_{K,L}(X)$;
3. If X is locally compact, then NE $\uparrow$ BM($C_k(X)$) iff K $\uparrow$ $G^o_{K,L}(X)$.

Theorem (McCoy, Ntantu)

1. If NE $\uparrow$ BM($C_k(X)$) then K $\uparrow$ $G^o_{K,L}(X)$;
2. If $C_k(X)$ is Baire, then L $\not\uparrow$ $G^o_{K,L}(X)$;
3. If X is locally compact, then NE $\uparrow$ BM($C_k(X)$) iff K $\uparrow$ $G^o_{K,L}(X)$.

Proof of(2) Suppose L $\uparrow$ $G^o_{K,L}(X)$

Theorem (McCoy, Ntantu)

1. If NE $\uparrow$ BM($C_k(X)$) then K $\uparrow$ $G^o_{K,L}(X)$;
2. If $C_k(X)$ is Baire, then L $\not\uparrow$ $G^o_{K,L}(X)$;
3. If X is locally compact, then NE $\uparrow$ BM($C_k(X)$) iff K $\uparrow$ $G^o_{K,L}(X)$.

Proof of(2) Suppose L $\uparrow$ $G^o_{K,L}(X)$
Claim. E $\uparrow$ BM($C_k(X)$) (so $C_k(X)$ not Baire, contradiction).

Theorem (McCoy, Ntantu)

1. If NE $\uparrow$ BM($C_k(X)$) then K $\uparrow$ $G^o_{K,L}(X)$;
2. If $C_k(X)$ is Baire, then L $\not\uparrow$ $G^o_{K,L}(X)$;
3. If X is locally compact, then NE $\uparrow$ BM($C_k(X)$) iff K $\uparrow$ $G^o_{K,L}(X)$.

Proof of(2) Suppose L $\uparrow$ $G^o_{K,L}(X)$
Claim. E $\uparrow$ BM($C_k(X)$) (so $C_k(X)$ not Baire, contradiction).
W.l.o.g., in the n^{th} round, Non-empty chooses a basic open set of the form $B(K_n, f_n, \epsilon_n) = \{g \in C_k(X) : \forall x \in K_n(|g(x) - f_n(x)| < \epsilon_n\}$, where K_n is compact.

Theorem (McCoy, Ntantu)

1. If NE $\uparrow$ BM($C_k(X)$) then K $\uparrow$ $G^o_{K,L}(X)$;
2. If $C_k(X)$ is Baire, then L $\not\uparrow$ $G^o_{K,L}(X)$;
3. If X is locally compact, then NE $\uparrow$ BM($C_k(X)$) iff K $\uparrow$ $G^o_{K,L}(X)$.

Proof of(2) Suppose L $\uparrow$ $G^o_{K,L}(X)$
Claim. E $\uparrow$ BM($C_k(X)$) (so $C_k(X)$ not Baire, contradiction).
W.l.o.g., in the n^{th} round, Non-empty chooses a basic open set of the form $B(K_n, f_n, \epsilon_n) = \{g \in C_k(X) : \forall x \in K_n(|g(x) - f_n(x)| < \epsilon_n\}$, where K_n is compact.

Let L_n be L's response to K_n in $G^o_{K,L}(X)$ using a winning strategy.

Theorem (McCoy, Ntantu)

1. If NE $\uparrow$ BM($C_k(X)$) then K $\uparrow G^o_{K,L}(X)$;
2. If $C_k(X)$ is Baire, then L $\not\uparrow G^o_{K,L}(X)$;
3. If X is locally compact, then NE $\uparrow$ BM($C_k(X)$) iff K $\uparrow G^o_{K,L}(X)$.

Proof of(2) Suppose L $\uparrow G^o_{K,L}(X)$
Claim. E $\uparrow$ BM($C_k(X)$) (so $C_k(X)$ not Baire, contradiction).
W.l.o.g., in the n^{th} round, Non-empty chooses a basic open set of the form $B(K_n, f_n, \epsilon_n) = \{g \in C_k(X) : \forall x \in K_n(|g(x) - f_n(x)| < \epsilon_n\}$, where K_n is compact.

Let L_n be L's response to K_n in $G^o_{K,L}(X)$ using a winning strategy.

Then *Empty* plays $B(K_n, f_n, \epsilon_n\} \cap B(L_n, c_n, 1/3)$, where c_n = constant n

Theorem (McCoy, Ntantu)

1. If NE $\uparrow$ BM($C_k(X)$) then K $\uparrow$ $G^o_{K,L}(X)$;
2. If $C_k(X)$ is Baire, then L $\not\uparrow$ $G^o_{K,L}(X)$;
3. If X is locally compact, then NE $\uparrow$ BM($C_k(X)$) iff K $\uparrow$ $G^o_{K,L}(X)$.

Proof of(2) Suppose L $\uparrow$ $G^o_{K,L}(X)$
Claim. E $\uparrow$ BM($C_k(X)$) (so $C_k(X)$ not Baire, contradiction).
W.l.o.g., in the n^{th} round, Non-empty chooses a basic open set of the form $B(K_n, f_n, \epsilon_n) = \{g \in C_k(X) : \forall x \in K_n(|g(x) - f_n(x)| < \epsilon_n\}$, where K_n is compact.

Let L_n be L's response to K_n in $G^o_{K,L}(X)$ using a winning strategy.

Then *Empty* plays $B(K_n, f_n, \epsilon_n\} \cap B(L_n, c_n, 1/3)$, where c_n = constant n
Suppose $\phi \in \bigcap_{n\in\omega} B(L_n, c_n, 1/3)$.

Theorem (McCoy, Ntantu)

1. If NE $\uparrow$ BM($C_k(X)$) then K $\uparrow$ $G^o_{K,L}(X)$;
2. If $C_k(X)$ is Baire, then L $\not\uparrow$ $G^o_{K,L}(X)$;
3. If X is locally compact, then NE $\uparrow$ BM($C_k(X)$) iff K $\uparrow$ $G^o_{K,L}(X)$.

Proof of(2) Suppose L $\uparrow$ $G^o_{K,L}(X)$
Claim. E $\uparrow$ BM($C_k(X)$) (so $C_k(X)$ not Baire, contradiction).
W.l.o.g., in the n^{th} round, Non-empty chooses a basic open set of the form $B(K_n, f_n, \epsilon_n) = \{g \in C_k(X) : \forall x \in K_n(|g(x) - f_n(x)| < \epsilon_n\}$, where K_n is compact.

Let L_n be L's response to K_n in $G^o_{K,L}(X)$ using a winning strategy.

Then *Empty* plays $B(K_n, f_n, \epsilon_n\} \cap B(L_n, c_n, 1/3)$, where c_n = constant n
Suppose $\phi \in \bigcap_{n \in \omega} B(L_n, c_n, 1/3)$. Then $\phi(L_n) \subset (n - 1/3, n + 1/3)$ for all n.

Theorem (McCoy, Ntantu)

1. If NE $\uparrow$ BM($C_k(X)$) then K $\uparrow$ $G^o_{K,L}(X)$;
2. If $C_k(X)$ is Baire, then L $\not\uparrow$ $G^o_{K,L}(X)$;
3. If X is locally compact, then NE $\uparrow$ BM($C_k(X)$) iff K $\uparrow$ $G^o_{K,L}(X)$.

Proof of(2) Suppose L $\uparrow$ $G^o_{K,L}(X)$
Claim. E $\uparrow$ BM($C_k(X)$) (so $C_k(X)$ not Baire, contradiction).
W.l.o.g., in the n^{th} round, Non-empty chooses a basic open set of the form $B(K_n, f_n, \epsilon_n) = \{g \in C_k(X) : \forall x \in K_n(|g(x) - f_n(x)| < \epsilon_n\}$, where K_n is compact.

Let L_n be L's response to K_n in $G^o_{K,L}(X)$ using a winning strategy.

Then *Empty* plays $B(K_n, f_n, \epsilon_n\} \cap B(L_n, c_n, 1/3)$, where c_n = constant n
Suppose $\phi \in \bigcap_{n\in\omega} B(L_n, c_n, 1/3)$. Then $\phi(L_n) \subset (n - 1/3, n + 1/3)$ for all n. Thus $\{L_n : n \in \omega\}$ has a discrete open expansion, contradiction.

Theorem (Ma, GG)

If X is locally compact, then $C_k(X)$ is Baire iff L $\not\uparrow G^{o}_{K,L}(X)$.

Theorem (Ma, GG)

If X is locally compact, then $C_k(X)$ is Baire iff L $\not\uparrow$ $G^o_{K,L}(X)$.

Question

Is it true that for any completely regular space X, $C_k(X)$ is Baire iff $L \not\uparrow G^o_{K,L}(X)$? That $NE \uparrow BM(C_k(X))$ iff $K \uparrow G^o_{K,L}(X)$?

A non-game theory characterization of "L has no winning strategy in $G^o_{K,L}(X)$"

A non-game theory characterization of "L has no winning strategy in $G^o_{K,L}(X)$"

Definition. A collection $\mathcal{L}$ of non-empty compact subsets of X is said to *move off* the compact sets if for every compact subset K of X, there is some $L \in \mathcal{L}$ with $K \cap L = \emptyset$.

A non-game theory characterization of "L has no winning strategy in $G^o_{K,L}(X)$"

Definition. A collection $\mathcal{L}$ of non-empty compact subsets of X is said to *move off* the compact sets if for every compact subset K of X, there is some $L \in \mathcal{L}$ with $K \cap L = \emptyset$.

The space X is said to have the *Moving Off Property (MOP)* iff every collection $\mathcal{L}$ which moves off the compact sets contains an infinite subcollection which has a discrete open expansion.

A non-game theory characterization of "L has no winning strategy in $G^o_{K,L}(X)$"

Definition. A collection $\mathcal{L}$ of non-empty compact subsets of X is said to *move off* the compact sets if for every compact subset K of X, there is some $L \in \mathcal{L}$ with $K \cap L = \emptyset$.

The space X is said to have the *Moving Off Property (MOP)* iff every collection $\mathcal{L}$ which moves off the compact sets contains an infinite subcollection which has a discrete open expansion.

Theorem

TFAE:

1. *X has the MOP;*
2. $L \not\uparrow G^o_{K,L}(X)$.

A non-game theory characterization of "L has no winning strategy in $G^o_{K,L}(X)$"

Definition. A collection $\mathcal{L}$ of non-empty compact subsets of X is said to *move off* the compact sets if for every compact subset K of X, there is some $L \in \mathcal{L}$ with $K \cap L = \emptyset$.

The space X is said to have the *Moving Off Property (MOP)* iff every collection $\mathcal{L}$ which moves off the compact sets contains an infinite subcollection which has a discrete open expansion.

Theorem

TFAE:

1. *X has the MOP;*
2. *$L \not\uparrow G^o_{K,L}(X)$.*

Question

Does X have MOP iff $C_k(X)$ Baire?

There is some subtlety in determining when X (even locally compact X) has the property "L has no winning strategy in $G_{K,L}(X)$":

There is some subtlety in determining when X (even locally compact X) has the property "L has no winning strategy in $G_{K,L}(X)$":
Consider spaces $T \cup A$, where T is the Cantor tree and A is a subset of the Cantor set.

There is some subtlety in determining when X (even locally compact X) has the property "L has no winning strategy in $G_{K,L}(X)$":
Consider spaces $T \cup A$, where T is the Cantor tree and A is a subset of the Cantor set.

Theorem (Ma)

The following are equivalent:

1. $C_k(T \cup A)$ is a Baire space;
2. L $\not\uparrow G_{K,L}(X)$
3. A is a γ-set.

There is some subtlety in determining when X (even locally compact X) has the property "L has no winning strategy in $G_{K,L}(X)$":
Consider spaces $T \cup A$, where T is the Cantor tree and A is a subset of the Cantor set.

Theorem (Ma)

The following are equivalent:

1. $C_k(T \cup A)$ is a Baire space;
2. L $\not\uparrow G_{K,L}(X)$
3. A is a γ-set.

$A \subset \mathbb{R}$ is a *γ-set* if, given any collection $\mathcal{U}$ of open sets such that any finite subset of A is contained in some member of $\mathcal{U}$, there are $U_0, U_1, \ldots$ in $\mathcal{U}$ such that $A \subset \bigcup_{n\in\omega} \bigcap_{i\geq n} U_i$.

Todorcevic showed that it is consistent for there to be two γ-sets A_0 and A_1 whose topological sum is not a γ-set.

Todorcevic showed that it is consistent for there to be two γ-sets A_0 and A_1 whose topological sum is not a γ-set.
Since $C_k(X_0 \oplus X_1) \cong C_k(X_0) \times C_k(X_1)$, Ma obtained the following corollary.

Todorcevic showed that it is consistent for there to be two γ-sets A_0 and A_1 whose topological sum is not a γ-set.
Since $C_k(X_0 \oplus X_1) \cong C_k(X_0) \times C_k(X_1)$, Ma obtained the following corollary.

Corollary

There are, consistently, two function spaces with the compact-open topology which are Baire but whose product is not.

Todorcevic showed that it is consistent for there to be two γ-sets A_0 and A_1 whose topological sum is not a γ-set.
Since $C_k(X_0 \oplus X_1) \cong C_k(X_0) \times C_k(X_1)$, Ma obtained the following corollary.

Corollary

There are, consistently, two function spaces with the compact-open topology which are Baire but whose product is not.

But we don't know about *ZFC* examples.

Question

Are there examples in ZFC of two Baire function spaces whose product is not Baire?

Some other games and/or applications

Tel'garsky(1975) The countably metacompact game $CM(X)$

Some other games and/or applications

Tel'garsky(1975) The countably metacompact game $CM(X)$

In the n^{th} round, I chooses closed $C_n \subset C_{n-1}$

Some other games and/or applications

Tel'garsky(1975) The countably metacompact game $CM(X)$

In the n^{th} round, I chooses closed $C_n \subset C_{n-1}$

II chooses open $U_n \supset C_n$

Some other games and/or applications

Tel'garsky(1975) The countably metacompact game $CM(X)$

In the n^{th} round, I chooses closed $C_n \subset C_{n-1}$

II chooses open $U_n \supset C_n$

I wins if $\bigcap_{n\in\omega} C_n = \emptyset$ but $\bigcap_{n\in\omega} U_n \neq \emptyset$

Some other games and/or applications

Tel'garsky(1975) The countably metacompact game $CM(X)$

In the n^{th} round, I chooses closed $C_n \subset C_{n-1}$

II chooses open $U_n \supset C_n$

I wins if $\bigcap_{n\in\omega} C_n = \emptyset$ but $\bigcap_{n\in\omega} U_n \neq \emptyset$

X not countably metacompact $\Rightarrow$ I $\uparrow CM(X)$

Some other games and/or applications

Tel'garsky(1975) The countably metacompact game $CM(X)$

In the n^{th} round, I chooses closed $C_n \subset C_{n-1}$

II chooses open $U_n \supset C_n$

I wins if $\bigcap_{n\in\omega} C_n = \emptyset$ but $\bigcap_{n\in\omega} U_n \neq \emptyset$

X not countably metacompact $\Rightarrow$ I $\uparrow CM(X)$

Theorem

$X \times M$ is normal for every metrizable space M iff X is normal and $II \uparrow CM(X)$.

Babinkostova (2005) The screening game $S(X)$
In the n^{th} round, I chooses open cover $\mathcal{U}_n$ of X

Babinkostova (2005) The screening game $S(X)$
In the n^{th} round, I chooses open cover $\mathcal{U}_n$ of X

II chooses disjoint open refinement $\mathcal{V}_n$ of $\mathcal{U}_n$

Babinkostova (2005) The screening game $S(X)$
In the n^{th} round, I chooses open cover $\mathcal{U}_n$ of X

II chooses disjoint open refinement $\mathcal{V}_n$ of $\mathcal{U}_n$

II $\uparrow S(X)$ if $\bigcup_{n\in\omega}\mathcal{V}_n$ covers X

Babinkostova (2005) The screening game $S(X)$

In the n^{th} round, I chooses open cover $\mathcal{U}_n$ of X

II chooses disjoint open refinement $\mathcal{V}_n$ of $\mathcal{U}_n$

II $\uparrow S(X)$ if $\bigcup_{n\in\omega} \mathcal{V}_n$ covers X

Theorem

(Babinkostova)

1. *II $\uparrow S(X)$ iff X is countable dimensional;*
2. *II has winning strategy in game of length $k+1$ iff X is $\leq k$ dimensional.*

A space X is *selectively separable (SS)* if, given dense sets $D_0, D_1, \ldots,$ there are finite $F_i \subset D_i$ with $\bigcup_{n\in\omega} F_n$ dense.

A space X is *selectively separable (SS)* if, given dense sets $D_0, D_1, \ldots,$ there are finite $F_i \subset D_i$ with $\bigcup_{n\in\omega} F_n$ dense.

The game SS(X) (Dow, Barman): In round n, I chooses dense D_n, II chooses finite $F_n \subset D_n$. II *wins* if $\bigcup_{n\in\omega} F_n$ is dense.

A space X is *selectively separable (SS)* if, given dense sets $D_0, D_1, \ldots,$ there are finite $F_i \subset D_i$ with $\bigcup_{n\in\omega} F_n$ dense.

The game SS(X) (Dow, Barman): In round n, I chooses dense D_n, II chooses finite $F_n \subset D_n$. II *wins* if $\bigcup_{n\in\omega} F_n$ is dense.

X is SS^+ if II $\uparrow SS(X)$.

A space X is *selectively separable (SS)* if, given dense sets $D_0, D_1, \ldots,$ there are finite $F_i \subset D_i$ with $\bigcup_{n\in\omega} F_n$ dense.

The game SS(X) (Dow, Barman): In round n, I chooses dense D_n, II chooses finite $F_n \subset D_n$. II *wins* if $\bigcup_{n\in\omega} F_n$ is dense.

X is SS^+ if II $\uparrow SS(X)$.

Countable π-base $\Rightarrow SS^+ \Rightarrow SS$

A space X is *selectively separable (SS)* if, given dense sets $D_0, D_1, \ldots$, there are finite $F_i \subset D_i$ with $\bigcup_{n\in\omega} F_n$ dense.

The game SS(X) (Dow, Barman): In round n, I chooses dense D_n, II chooses finite $F_n \subset D_n$. II *wins* if $\bigcup_{n\in\omega} F_n$ is dense.

X is SS^+ if II $\uparrow SS(X)$.

Countable π-base $\Rightarrow SS^+ \Rightarrow SS$
Separable Fréchet $\Rightarrow$ SS

A space X is *selectively separable (SS)* if, given dense sets $D_0, D_1, \ldots$, there are finite $F_i \subset D_i$ with $\bigcup_{n\in\omega} F_n$ dense.

The game SS(X) (Dow, Barman): In round n, I chooses dense D_n, II chooses finite $F_n \subset D_n$. II *wins* if $\bigcup_{n\in\omega} F_n$ is dense.

X is SS^+ if II $\uparrow SS(X)$.

Countable π-base $\Rightarrow SS^+ \Rightarrow SS$
Separable Fréchet $\Rightarrow$ SS
$SS \not\Rightarrow SS^+$.

Theorem

(Dow) X countable $SS^+ \Rightarrow$ II has Markov winning strategy in $SS(X)$.

So, for each dense D, for each $n \in \omega$, one can assign finite $F(D, n) \subset D$ such that, if $D_0, D_1, \ldots$ are dense, then $\bigcup_{n\in\omega} F(D_n, n)$ is dense.

Idea of proof.
Let σ be winning strategy for II.

Idea of proof.

Let σ be winning strategy for II.

X countable $\Rightarrow$ II has only countably many possible replies

Idea of proof.
Let σ be winning strategy for II.

X countable $\Rightarrow$ II has only countably many possible replies

For each possible first round reply F, choose $D(F)$ dense such that $\sigma(D(F)) = F$.

Idea of proof.

Let σ be winning strategy for II.

X countable $\Rightarrow$ II has only countably many possible replies

For each possible first round reply F, choose $D(F)$ dense such that $\sigma(D(F)) = F$.

For each possible second round reply F' of II, choose dense $D(F, F')$ such that $\sigma(D(F), D(F, F')) = F'$.

Idea of proof.
Let σ be winning strategy for II.

X countable $\Rightarrow$ II has only countably many possible replies

For each possible first round reply F, choose $D(F)$ dense such that $\sigma(D(F)) = F$.

For each possible second round reply F' of II, choose dense $D(F, F')$ such that $\sigma(D(F), D(F, F')) = F'$.
Etc.

Idea of proof.
Let σ be winning strategy for II.

X countable $\Rightarrow$ II has only countably many possible replies

For each possible first round reply F, choose $D(F)$ dense such that $\sigma(D(F)) = F$.

For each possible second round reply F' of II, choose dense $D(F, F')$ such that $\sigma(D(F), D(F, F')) = F'$.
Etc.
This constructs a (countable) tree of finite sequences of dense sets. Let $t_0, t_1, \ldots$ be the nodes of the tree. The Markov winning strategy for II is:

Idea of proof.

Let σ be winning strategy for II.

X countable $\Rightarrow$ II has only countably many possible replies

For each possible first round reply F, choose $D(F)$ dense such that $\sigma(D(F)) = F$.

For each possible second round reply F' of II, choose dense $D(F, F')$ such that $\sigma(D(F), D(F, F')) = F'$.
Etc.
This constructs a (countable) tree of finite sequences of dense sets. Let $t_0, t_1, \dots$ be the nodes of the tree. The Markov winning strategy for II is:

Given dense D in round n, II plays $\sigma(t_n {}^\frown \langle D \rangle)$.

Idea of proof.

Let σ be winning strategy for II.

X countable $\Rightarrow$ II has only countably many possible replies

For each possible first round reply F, choose $D(F)$ dense such that $\sigma(D(F)) = F$.

For each possible second round reply F' of II, choose dense $D(F, F')$ such that $\sigma(D(F), D(F, F')) = F'$.
Etc.
This constructs a (countable) tree of finite sequences of dense sets. Let $t_0, t_1, \ldots$ be the nodes of the tree. The Markov winning strategy for II is:
Given dense D in round n, II plays $\sigma(t_n^\frown \langle D \rangle)$.

[Same result for any game with II having only countably many responses, and I's legal plays unchanged during the game.]

References and further reading

SURVEYS

G. Gruenhage, *The story of a topological game*, Rocky Mountain Journal Math. 36 (2006), no. 6, 1885–1914.

M. Scheepers, *Topological games*, in: Encyclopedia of General Topology, K.P. Hart, J. Nagata, and J.E. Vaughan, eds., Elsevier, Amsterdam, 2004, 438-442.

R. Telgarsky, *Topological games: On the 50th anniversary of the Banach-Mazur game*, Rocky Mountain Journal Math. 17(1987), 227-276.
Y. Yajima, *Topological games and applications*, in: Topics in general topology, North-Holland, Amsterdam (1989), 523–562.

OTHER REFERENCES

L. Babinkostova, *Selective screenability game and covering dimension*, Top. Proc. 29 (2005), 13-17.

A. Dow and D. Barman, *Selective separability and SS^+*, Topology Proc., to appear.

F. Galvin and R. Telgarsky, *Stationary strategies in topological games*, Topology Appl. 22(1986), 51-69.

G. Gruenhage, *Covering properties on $X^2 \setminus \Delta$, W-sets, and compact subsets of Σ-products*, Topology Appl. 17 (1984), no. 3, 287–304.

G. Gruenhage, *Games, covering properties, and Eberlein compacts*, Topology Appl. 23(1986), 291-297.

W. Hurewicz, *Uber eine Verallgemeinerung des Borelschen Theorems*, Math. Zeitschrift 24(1926), 401-421.

J. C. Oxtoby, *The Banach-Mazur game and Banach category theorem, Contribution to the theory of games*, Annals of Math. Studies 39 (1957), 157-163.

R. Telgarsky, *Spaces defined by topological games*, Fund. Math. 88(1975), 193-223.