The Logic of Stone Spaces

John Harding and Guram Bezhanishvili

New Mexico State University
www.math.nmsu.edu/JohnHarding.html

jharding@nmsu.edu

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Basics

$\mathbf{CL}$ = the variety of all closure algebras (B, C)

$X^* = (\mathcal{P}X, C)$ where X is a topological space

View subvarieties of $\mathbf{CL}$ as extensions of Lewis' **S4**

- $\mathbf{S4} \leftrightarrow \mathbf{CL}$
- $\mathbf{S4.1} \leftrightarrow \mathbf{CL} + \mathrm{IC}x \leq \mathrm{CI}x$
- $\mathbf{S4.2} \leftrightarrow \mathbf{CL} + \mathrm{CI}x \leq \mathrm{IC}x$
- etc.

Theorem (McKinsey-Tarski) If X is metrizable and has no isolated points, then X^* generates $\mathbf{CL}$.

Aim

For a Boolean algebra B with Stone space X, to determine the subvariety of **CL** generated by X^*, i.e. the modal logic of X. We can do this if B is complete or if B is countable.

Note For B countable and free, X is the Cantor space, so by the McKinsey-Tarski theorem its logic is **S4**.

Tools

Each quasiorder Q is a topological space where opens $:=$ upsets.

Many subvarieties of **CL** are generated by classes of quasiorders.

- **S4** by finite quasitrees.
- **S4**.1 by finite quasitrees with top level simple nodes.
- **S4**.2 by the $Q \oplus C$ with Q finite quasitree and C cluster.

$$X \xrightarrow{f} \!\!\!\!\rightarrow Y \quad \underbrace{\mathrm{cont} + \mathrm{open}}_{\mathrm{interior}} + \mathrm{onto} \quad \Rightarrow \quad Y^* \overset{f^{-1}}{\longmapsto} X^* \text{ } \mathbf{CL}\text{-embedding.}$$

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Example To show the logic of X is **S4**

Enough to find an onto interior $X \xrightarrow{f} Q$ for each finite quasitree Q as X^* will contain a generating set for **S4**.

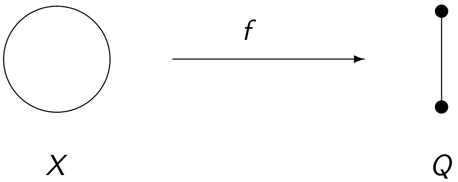
Tools

Our job amounts to finding interior onto maps $X \xrightarrow{f} Q$.

Lets look at some easy examples ...

Easiest example

For X the Stone space of B, when is there an interior onto map



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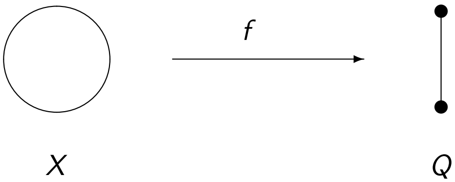


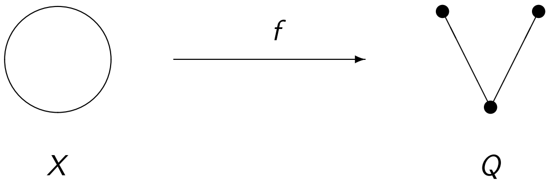
When X has a proper dense open set U $(= f^{-1}[\text{top}])$.

When B has a proper ideal whose join is 1.

When B is infinite.

Next easiest example

For X the Stone space of B, when is there an interior onto map



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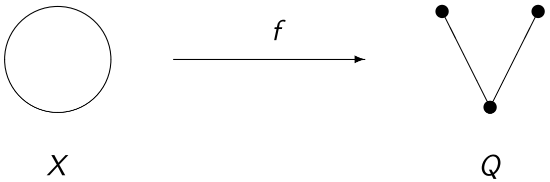


When X has disjoint regular open U, V with $U \cup V$ proper dense.

When B has a non-principal normal ideal.

When B is incomplete.

The logic of ω^*

$$\beta\omega = \text{the Stone Cech compactification of } \omega$$
$$\omega^* = \text{the remainder } \beta\omega - \omega$$
$$\omega^* = \text{the Stone space of } \mathcal{P}\omega/\mathit{Fin}.$$

Theorem The logic of ω^* is **S4**.

Proof. We need an interior onto map $\omega^* \xrightarrow{f} Q$ for each finite quasitree Q. For this we need a technical result to recursively build a tree of ideals in our Boolean algebra.

Lemma $(\mathfrak{a} = 2^\omega)$. For P a partition of $b \in \mathcal{P}\omega/\mathit{Fin}$ and $m \geq 1$, there are sets $P_1, \ldots, P_m$ and maps $f_1, \ldots, f_m$ with

1. $P_1 \cup \cdots \cup P_m = P$ and $P_i \cap P_j = \emptyset$ for each $i \neq j$.
2. $f_i : \mathit{Infinite}(P) \to P_i$ is 1-1 for each $i \leq m$.
3. $f_i(c) \in \mathit{Support}_P(c)$ for each $c \in \mathit{Infinite}(P)$ and each $i \leq m$.

Note $(\mathfrak{a} = 2^\omega)$ is an additional assumption of set theory.

Note We use this to recursively build a tree of ideals.

Corollaries

Theorem The logic of $\beta\omega$ is $\mathbf{S4.1.2}$.

Proof. Any interior $\omega^* \longrightarrow\!\!\!\!\!\rightarrow Q$ lifts to an interior $\beta\omega \longrightarrow\!\!\!\!\!\rightarrow Q \oplus 1$ and this is exactly what we need.

Theorem For B a complete Boolean algebra with Stone space X.

1. If B is finite, the logic of X is classical.
2. If B is infinite and atomic, the logic of X is $\mathbf{S4.1.2}$.
3. Otherwise the logic of X is $\mathbf{S4.2}$.

Proof. Such X has a closed subspace homeomorphic to $\beta\omega$. We use this to build our map $X \longrightarrow\!\!\!\!\!\rightarrow Q \oplus C$ for the difficult case 3.

Countable Boolean algebras

For B Boolean with Stone space X the following are equivalent

- B is countable
- B is generated by a countable chain C
- X is metrizable

The atomless case gives **S4** by McKinsey-Tarski.

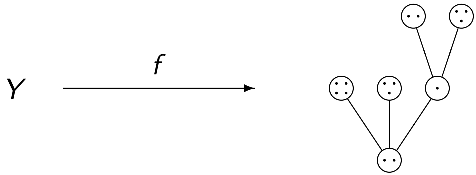
The scattered case gives $\mathbf{Grz_n}$ for some $n \leq \omega$ by old results.

So we may assume B is generated by a chain C where each interval contains a cover, and the condensation D of C is $\mathbb{Q}$. We will show **S4.1** is the logic in this case.

Our setup ...

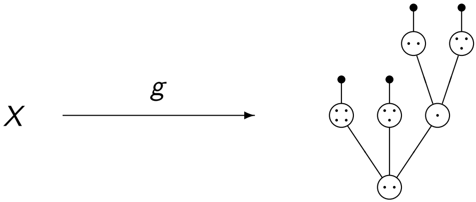
$$
\begin{aligned}
D &= \text{condensation of } C \\
Y &= \text{Stone space of free Boolean ext of } D \text{ (so } Y \simeq \text{Cantor)} \\
Y &\leq X
\end{aligned}
$$

Lets sketch the idea ...

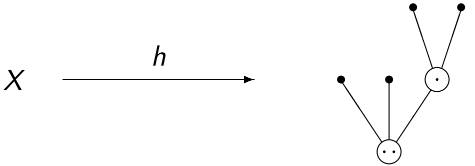


We get this as $Y \simeq$ Cantor

The hard part is to use the way Y sits in X to extend to ...



As squishing the top parts is interior we get



The Q we can get on the right are the ones we need to show **S4.1**.

Questions

Is the assumption ($\mathfrak{a} = 2^\omega$) necessary for the ω^* result?

Extend countable results to any B generated by a chain, or tree.

Conjecture

The varieties generated by X^* for a Stone space X are exactly the finite joins of the ones above.

Little question

Does every atomless B have a dense ideal I with B/I atomless?