Modal Logics of Some Subspaces of Rational Numbers: Diamond as Derivative

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McKinsey and Tarski 1944

- 2 Topological Interpretations for $\Diamond$

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- *Main Result:* The c-logic of a dense-in-itself metrizable space is **S4**.

More Background

Diamond as Derivative

- **Esakia:**
  (1981) **GL** is the d-logic of all scattered spaces.
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- **Bezhanishvili:**
  (2010 w/ Morandi) $\mathbf{GL}_n$ is the d-logic of any ordinal α where $\omega^{n-1} < \alpha \leq \omega^n$.
  (2010 w/ Esakia and Gabelaia) **K4** is the d-logic of all Stone spaces.

Essential Preliminaries:

Syntax

- **Alphabet**
  Propositional Variables: $\mathfrak{Var} = \{p_0,\ p_1,\ p_2,\ \ldots\}$
  Binary Connectives: conjunction $\wedge$ and disjunction $\vee$
  Unary Connectives: negation $\neg$, box $\Box$, and diamond $\Diamond$

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- **Well Formed Formulas:** $\mathfrak{Form}$

$$\mathfrak{Var} \subseteq \mathfrak{Form}$$

and if $\varphi, \psi \in \mathfrak{Form}$, then

$$(\varphi \wedge \psi), \; (\varphi \vee \psi), \; \neg\varphi, \; \Box\varphi, \; \Diamond\varphi \in \mathfrak{Form}$$

Essential Preliminaries:

The Logics of Interest 1

- **Modal Logic: L**

  L $\subseteq \mathfrak{Form}$, containing all substitution instances of classical tautologies, the formulas

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \tag{K}$$

$$\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi \tag{dual}$$

  and which is closed under modus ponens and $\Box$-necessitation

$$\frac{\varphi, \varphi \to \psi}{\psi} \text{ and } \frac{\varphi}{\Box\varphi}$$

  least: **K**

Essential Preliminaries:

The Logics of Interest 2

- **Specific Logics:**

$$
\begin{aligned}
\mathbf{K4} &= \mathbf{K} + \Diamond\Diamond\varphi \to \Diamond\varphi \\
\mathbf{KD4} &= \mathbf{K4} + \Diamond\top \\
\mathbf{GL} &= \mathbf{K} + \Box(\Box\varphi \to \varphi) \to \Box\varphi \\
\mathbf{GL}_n &= \mathbf{GL} + \Box^n\bot
\end{aligned}
$$

Main Technique

Utilize results in Kripke semantics and transfer to d-semantics

- **Main Tool:** *d-morphism*–a map $f : (X, \tau) \rightarrow (W, R)$ so that for any $A \subseteq W$

$$f^{-1}\left(R^{-1}(A)\right) = d\left(f^{-1}(A)\right)$$

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    then $(X, \tau) \vDash \varphi$ implies $(W, R) \vDash \varphi$;
    or equivalently $(W, R) \nvDash \varphi$ implies $(X, \tau) \nvDash \varphi$
    (2005 Bezhanishvili, Esakia, and Gabelaia)

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Logic	**Class**
K4	countable irreflexive tress
KD4	the irreflexive ω-branching ω-tall tree, $\mathcal{T}_\omega$
GL	finite irreflexive trees
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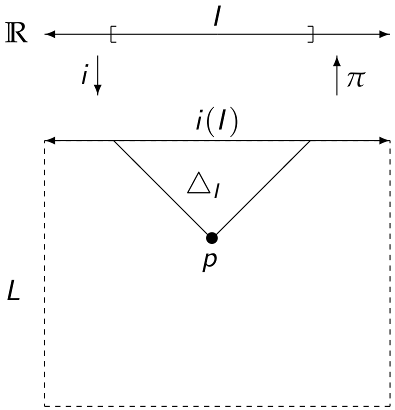
- **Observation:** all classes consist of countable irreflexive trees

Construction

Theorem: For any countable irreflexive tree, $\mathcal{T}$, there is, $Q_{\mathcal{T}}$, a subspace of $\mathbb{Q}$ and a function $f : Q_{\mathcal{T}} \to \mathcal{T}$ that is a surjective d-morphism.

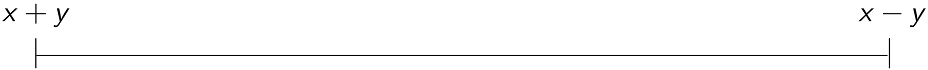
Let $L = \mathbb{R} \times (-\infty, 0]$ be 'the lower half plane' of $\mathbb{R}^2$



Construction

Dissecting an interval

From $(x, y) \in L = \mathbb{R} \times (-\infty, 0]$ with $y \neq 0$, we get the interval $[x + y, x - y]$



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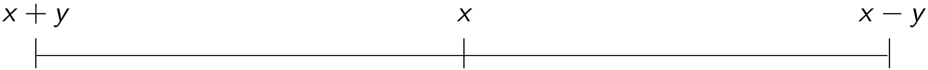

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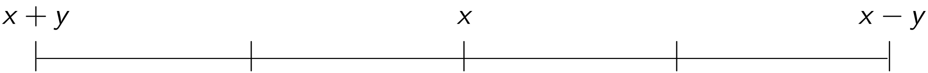

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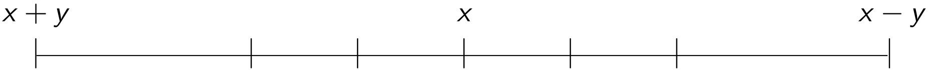

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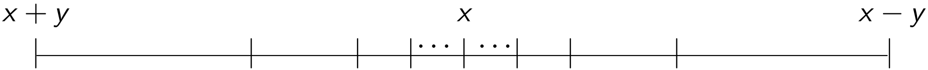

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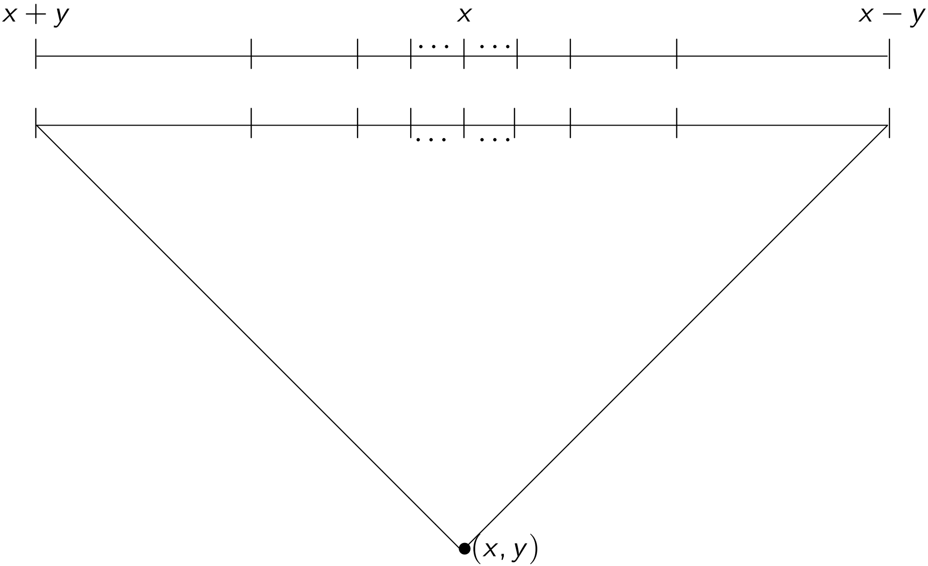

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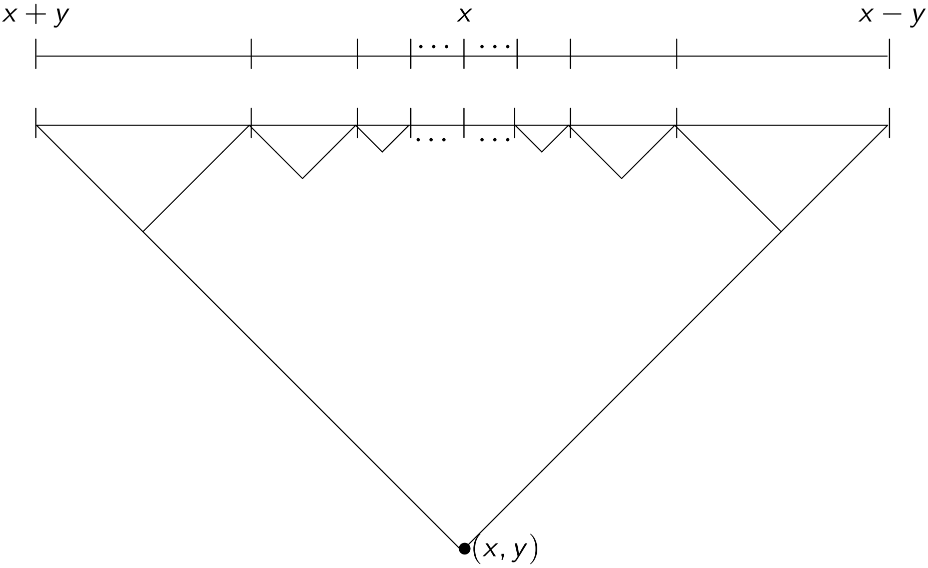

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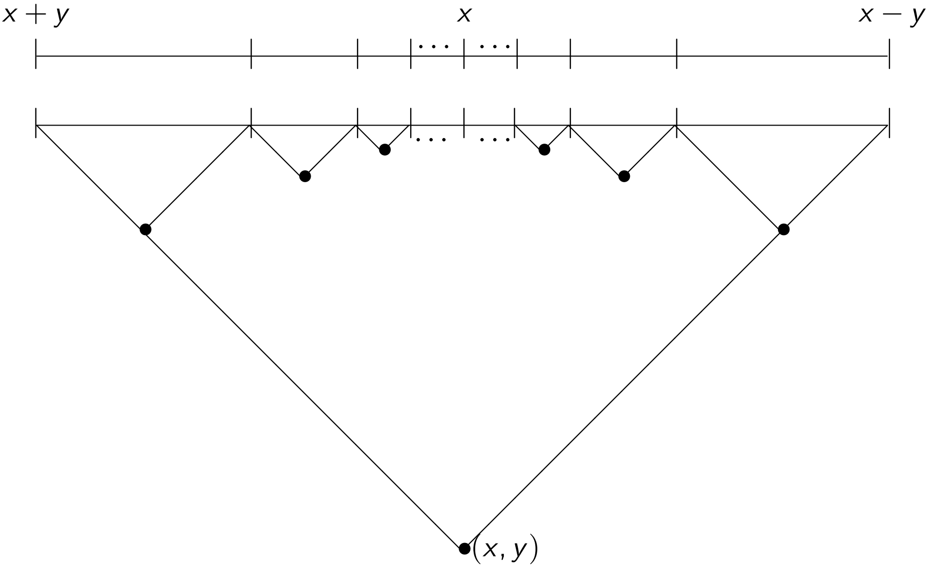

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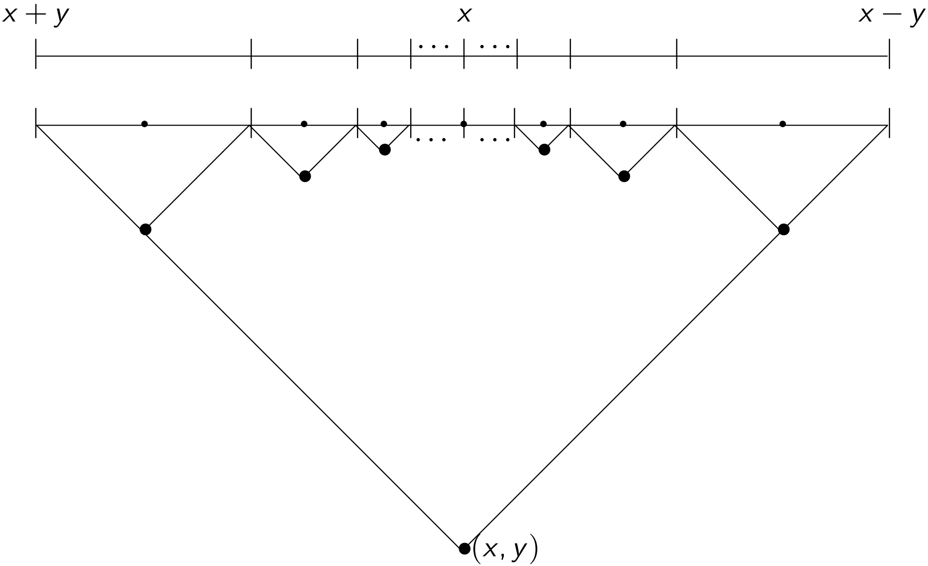

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  'Proof': $\forall \varphi \notin \mathbf{L}$, $\exists \mathcal{T}$ a tree with $\mathcal{T} \models \mathbf{L}$ and $\mathcal{T} \not\models \varphi$. The countable disjoint union of $\mathbb{Q}$ is homeomorphic to $\mathbb{Q}$.

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  - $Var\left(\mathcal{P}\left(Q_{\mathbf{K4}}\right), d\right)$ is defined by equations MA1, MA2 and **4**.

Pushing Further:

The Universal Modality

Add to the language the Universal Modality:

$$U \text{ (box-like) and } E\text{(diamond-like)}$$

- Semantics: in any model (topological or Kripke)

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- Global Expressive Power
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  - Connected Topological Space

Minimal Extensions

Let **L** be a unimodal logic. The minimal extension of **L**, denoted **L**.**U**, is a bimodal logic extending **L** containing

$$\begin{array}{ll} U\varphi \to \varphi & U\varphi \to \Box\varphi \text{ (bridge axiom)} \\ U\varphi \to UU\varphi & U(\varphi \to \psi) \to (U\varphi \to U\psi) \\ \varphi \to UE\varphi & U\varphi \leftrightarrow \neg E \neg \varphi \end{array}$$

closed under modus ponens and U-necessitation, $\frac{\varphi}{U\varphi}$.

Results for Kripke Frames

Theorem: The following logics are defined by the indicated classes of forests (a type of Kripke frame):

Logic	Class of *finite* disjoint Unions of
K4.U	countable irreflexive trees
KD4.U	$\mathcal{T}_\omega$
GL.U	finite irreflexive trees
$\mathbf{GL}_n.\mathbf{U}$	finite irreflexive trees of depth $\leq n$

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- **Theorem:** For each $\mathbf{L} \in \{\mathbf{K4}, \mathbf{GL}, \mathbf{GL}_n\}$ there is a countable class of subspaces of $\mathbb{Q}$, $\mathcal{C}_{\mathbf{L}}$, so that the ud-logic of $\mathcal{C}_{\mathbf{L}}$ is $\mathbf{L}$.

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- **Note:** This construction does not provided $\mathcal{C}_{\mathbf{L}}$ to be a singleton except in the case for $\mathbf{KD4.U}$.

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  But soundness is checked case by case
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Thank You

Any Questions?