Lattice valued identities

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μ_p is the ***p*-cut**, a **cut set** or simply a **cut** of μ.

A p-cut of μ is the inverse image of the principal filter in L generated by p :

$$\mu_p = \mu^{-1}(\uparrow p).$$

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Proposition

For every $x \in X$,

$$\bigcap\{\mu_p \mid x \in \mu_p\} \in \mu_L.$$

Theorem

The collection μ_L of cuts of $\mu : X \to L$ is a complete lattice under inclusion.

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Synthesis by cuts

Theorem

If $\mu : X \to L$ is an L-valued function on X, then for every $x \in X$

$$\mu(x) = \bigvee \{p \in L \mid x \in \mu_p\}.$$

Representation theorem

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Theorem

Let X be a nonempty set and F a family of its subsets closed under arbitrary intersections and containing X (a closure system on X). Let also L be the lattice dual to $(F, \subseteq)$ and $\mu : X \to L$ an L-valued function on X defined by

$$\mu(x) := \bigcap \{f \in F \mid x \in f\}.$$

Then, the lattice of cut subsets of μ is isomorphic with $(F, \subseteq)$, and every $f \in F$ coincides with the corresponding cut μ_f.

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By the definition, L_μ consists of particular collections of images of μ in L and is a poset under inclusion.

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Theorem

L_μ is a lattice isomorphic with the lattice μ_L of cuts of μ, under

$$f : \mu_p \mapsto \uparrow p \cap \mu(X).$$

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$\mu \sim \nu$ if and only if the correspondence $f : \mu(x) \mapsto \nu(x)$, $x \in X$ is a bijection from $\mu(X)$ onto $\nu(X)$ which has an extension to an isomorphism from the lattice L_μ onto the lattice L_ν, given by the mapping

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If $\mu \sim \nu$, then the L-valued functions μ and ν on X are said to be **equivalent**.

Classification of functions in L^X

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Let $\mu, \nu : X \to L$. Then $\mu \sim \nu$ if and only if L-valued functions μ and ν have equal collections of cuts.

Example

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$$\mu = \begin{pmatrix} x & y & z \\ p & q & r \end{pmatrix} \qquad \nu = \begin{pmatrix} x & y & z \\ p & q & t \end{pmatrix} \qquad \pi = \begin{pmatrix} x & y & z \\ p & r & t \end{pmatrix}$$

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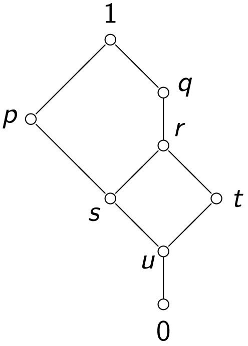
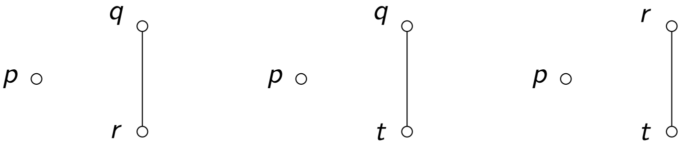
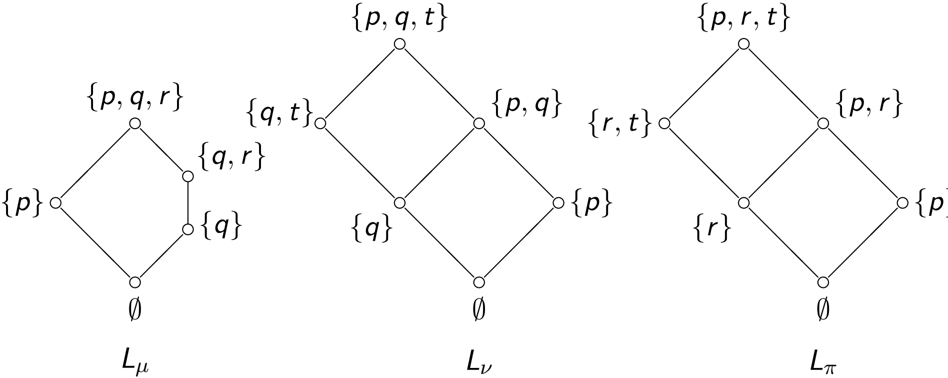


L

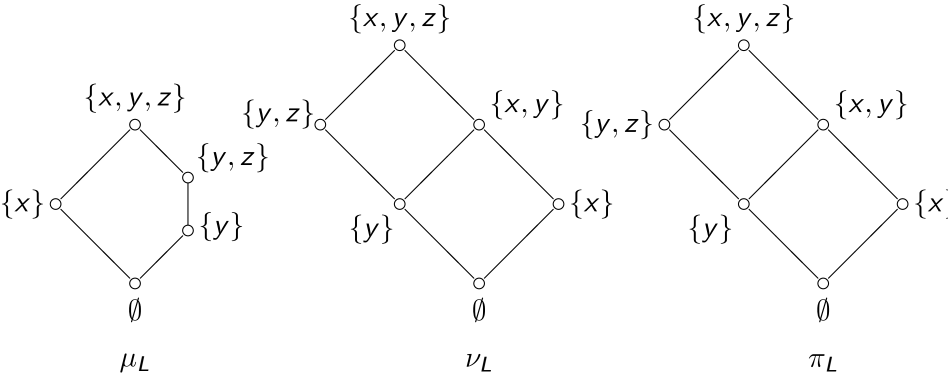
Lattice valued functions
Cuts applied



$\mu(X)$ $\nu(X)$ $\pi(X)$



$$L_\mu = (\{\uparrow p \cap \mu(X) \mid p \in L\}, \subseteq)$$



$$\mu_p = \mu^{-1}(\uparrow p); \quad \mu_L = \{\mu_p \mid p \in L\}$$

Representation of lattices

Representation of lattices

Theorem

Let L be a lattice of finite length, and X the set of its meet irreducible elements. Then there is an L-valued function $\mu : X \to L$ such that L is isomorphic with the dual of the lattice μ_L of cuts of μ, under $p \mapsto \mu_p$.

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An L-valued relation R on X is
reflexive and **symmetric** if it fulfills the analogue properties for its characteristic function; it is **transitive** if
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An L-valued relation R on X is a **lattice valued equivalence relation** on X if it is reflexive, symmetric and transitive.

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For a nullary operation (constant) $c \in F$,

$$\mu(c) = 1,$$

where 1 is the top element in L.

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If $\mu : A \to L$ is a lattice valued subalgebra of an algebra $\mathcal{A}$, then for every $p \in L$, the cut set μ_p is a subalgebra of $\mathcal{A}$.

Theorem

Let $\mathcal{A}$ be an algebra and $\mathcal{F}$ a collection of its subuniverses closed under arbitrary intersections and containing A. Let also L be the lattice dual to $(\mathcal{F}, \subseteq)$ and $\mu : A \to L$ an L-valued set on A defined by

$$\mu(x) := \bigcap\{B \in \mathcal{F} \mid x \in B\}.$$

Then, μ is an L-valued subalgebra of $\mathcal{A}$. In addition, the lattice of cut subalgebras of μ is isomorphic with $(\mathcal{F}, \subseteq)$, and every subalgebra $B \in \mathcal{F}$ coincides with the corresponding cut μ_B.

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Let $\mathcal{A} = (A, F)$ be an algebra and L a complete lattice, and $R : A^2 \to L$ be an L-valued relation on A.

R is said to be **compatible** with operations on $\mathcal{A}$ if for any (n-ary) $f \in F$ and all $x_1, \ldots, x_n, y_1, \ldots, y_n \in A$, we have that

$$\bigwedge_{i=1}^{n} R(x_i, y_i) \leqslant R(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)).$$

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An L-valued equivalence relation on $\mathcal{A}$ which is compatible with all operations is a **lattice valued congruence** relation on $\mathcal{A}$.

Theorem

If $R : A^2 \to L$ is a lattice valued congruence on an algebra $\mathcal{A}$, then for every $p \in L$, the cut relation R_p is a congruence on $\mathcal{A}$.

Let A be a nonempty set, L a complete lattice and $\mu : A \to L$ an L-valued set on A. An L-valued relation $\rho : A^2 \to L$ on A is said to be an ***L*-valued relation on μ** if for all $x, y \in A$

$$\rho(x, y) \leqslant \mu(x) \wedge \mu(y). \tag{1}$$

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$$\rho(x, y) \leqslant \mu(x) \wedge \mu(y). \tag{1}$$

Due to this boundary condition, we have the following definition. An L-valued relation ρ on an L-valued set μ is **reflexive** if for all $x, y \in A$,

$$\rho(x, x) = \mu(x). \tag{2}$$

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Obviously, by (1), a reflexive relation ρ on μ fulfils the following:

$$\text{For all } \ x, y \in A, \ \rho(x, x) \geqslant \rho(x, y) \ \text{ and } \ \rho(x, x) \geqslant \rho(y, x). \tag{3}$$

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An L-valued equivalence relation ρ on μ, fulfilling

$$\text{For all } \; x, y \in A, \; \rho(x,x) > \rho(x,y) \; \text{ and } \; \rho(x,x) > \rho(y,x), \tag{4}$$

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is called an ***L*-valued equality** relation on an L-valued set μ.

In addition, an L-valued relation ρ on μ is compatible with the operations on this L-valued subalgebra if for any (n-ary) $f \in F$ and all $x_1, \ldots, x_n, y_1, \ldots, y_n \in A$, we have that

$$\bigwedge_{i=1}^{n} R(x_i, y_i) \leqslant R(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)).$$

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Denote by $\mathrm{LCon}\,\mu$ and $\mathrm{LEq}\mu$ the collections of all L-valued congruences and all compatible L-valued equalities (respectively) on an L-valued subalgebra μ of an algebra $\mathcal{A}$. These can be naturally ordered by componentwise order $\leqslant$, inherited from L.

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Theorem

The poset $(\operatorname{LCon}\mu, \leqslant)$ *is a complete lattice, and the poset* $(\operatorname{LEq}\mu, \leqslant)$ *is a meet-semilattice, a semi-ideal in the former.*

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We say that $\rho : A^2 \longrightarrow L$ is an ***L*-valued weakly reflexive** relation on $\mathcal{A}$, if

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An L-valued relation $\rho : A^2 \to L$ on an algebra $\mathcal{A}$ which is weakly reflexive, symmetric and transitive, is called a **weak L-valued equivalence** on $\mathcal{A}$.

We have an algebra $\mathcal{A} = (A, F)$ and L-valued relations on it, i.e., mappings from A^2 to a complete lattice L.
We say that $\rho : A^2 \to L$ is an **L-valued weakly reflexive** relation on $\mathcal{A}$, if

$$\rho(c, c) = 1 \text{ for every constant } c \in F. \tag{5}$$

An L-valued relation $\rho : A^2 \to L$ on an algebra $\mathcal{A}$ which is weakly reflexive, symmetric and transitive, is called a **weak L-valued equivalence** on $\mathcal{A}$.
If, in addition, ρ fulfills also the condition

$$\text{For all } x, y \in A,\ \rho(x, x) > \rho(x, y) \text{ and } \rho(x, x) > \rho(y, x),$$

then ρ is a **weak L-valued equality** on $\mathcal{A}$.

For compatible weak L-valued equivalences we use the name **weak L-valued congruences** on $\mathcal{A}$. A subclass of weak L-valued congruences are **compatible weak L-valued equalities**.

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Theorem

If $\rho : A^2 \longrightarrow L$ is a weak L-valued congruence on an algebra $\mathcal{A}$, then the mapping $\mu_\rho : A \longrightarrow L$, defined by

$$\mu_\rho(x) := \rho(x, x) \tag{6}$$

is an L-valued subalgebra of $\mathcal{A}$.

The previous theorem gives a link between L-valued congruences on L-valued subalgebras and weak L-valued congruences on the whole algebra.

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Theorem

A weak L-valued congruence $\rho : A^2 \rightarrow L$ on an algebra $\mathcal{A}$ is an L-valued congruence on the L-valued subalgebra μ_ρ of A. Conversely, an L-valued congruence ρ on an L-valued subalgebra μ of $\mathcal{A}$ is a weak L-valued congruence on the whole algebra $\mathcal{A}$.

Theorem

The collection of all weak L-valued congruences on an algebra $\mathcal{A}$ is a complete lattice. Its sublattice of diagonal relations is isomorphic to the lattice of all L-valued subalgebras of $\mathcal{A}$. The lattice of L-valued congruences on each L-valued subalgebra of $\mathcal{A}$ is an interval sublattice.

Let $\mathcal{A} = (A, F)$ be an algebra and L a complete lattice.

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Let μ be an L-valued subalgebra of $\mathcal{A}$ and $E : A^2 \to L$ an L-valued equality on μ.

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If t_1, t_2 are terms in the language of $\mathcal{A}$, we consider the expression $E(t_1, t_2)$ as an **L-valued identity with respect to E**, or (briefly) **L-valued identity**, if E is fixed.

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Suppose that $x_1, \ldots, x_n$ are variables appearing in terms t_1, t_2. We say that an L-valued subalgebra μ of $\mathcal{A}$ **satisfies** the L-valued identity $E(t_1, t_2)$ (or that this L-valued identity is valid on L-valued subalgebra μ) if for all $x_1, \ldots, x_n \in A$

$$\bigwedge_{i=1}^{n} \mu(x_i) \leqslant E(t_1, t_2). \tag{7}$$

Proposition

Let $\mu : A \to L$ be an L-valued subalgebra of an algebra $\mathcal{A}$ and $E : A^2 \to L$ an L-valued equality on μ. If μ satisfies an L-valued identity $E(f,g)$, then also μ satisfies the identity $E_1(f,g)$, for every L-valued equality E_1 on μ, such that $E \leqslant E_1$.

Proposition

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Lemma

Let $\mu : A \to L$ be an L-valued subalgebra of an algebra $\mathcal{A}$, $\{E_i : A^2 \to L, i \in I\}$ a family of L-valued equalities on μ, and f, g terms in the language of $\mathcal{A}$. Now, if μ satisfies the identity $E_i(f,g)$ for every $i \in I$, then μ also satisfies the identity $E(f,g)$, where $E = \bigwedge_{i \in I} E_i$.

Corollary

If an L-valued subalgebra μ of $\mathcal{A}$ satisfies the identity $E(f,g)$ for an L-valued equality E, then there is the least L-valued equality on μ, denoted by $E_{\mu(f,g)}$, such that μ satisfies $E_{\mu(f,g)}(f,g)$.

Corollary

If an L-valued subalgebra μ of $\mathcal{A}$ satisfies the identity $E(f,g)$ for an L-valued equality E, then there is the least L-valued equality on μ, denoted by $E_{\mu(f,g)}$, such that μ satisfies $E_{\mu(f,g)}(f,g)$.

Corollary

Let μ be an L-valued subalgebra of an algebra $\mathcal{A}$. Then $\mathcal{A}$ satisfies the identity $f = g$ if and only if μ satisfies the identity $E(f,g)$ for every $E \in \mathrm{fEq}\,\mu$.

Corollary

Let μ be an L-valued subalgebra of an algebra $\mathcal{A}$ and L a complete lattice. Let also f, g be terms in the language of $\mathcal{A}$. Then the following hold.
The cut subalgebra μ_p of μ for $p \in L$ satisfies the identity $f = g$ if and only if the cut-relation $(E_{\mu(f,g)})_p$ of the least equality $E_{\mu(f,g)}$ is the ordinary equality on μ_p.

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Thank you for your attention!