Non-continuous satisfaction of identities

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BLAST, Boulder, Colorado

June 2, 2010

Four sections of the talk.

Basics

More nearly precise definitions.

An example: $\mu_1(A, \Sigma) = 0$; $\mu_2(A, \Sigma) = \mathrm{diam}(A)$.

Some further results

Compatibility: $A \models \Sigma$

Given a *topological space* A and a set Σ of equations in operation symbols F_t, we write

$$A \models \Sigma,$$

and say that A and Σ are *compatible*, iff there exist *continuous* operations $\overline{F_t}$ on A satisfying Σ.

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Examples: Groups on S^1, S^3 and $\mathbb{R}$, various matrix groups, many H-spaces, a lattice on $[0,1]$, a ternary median operation on Y, simple Σ on absolute-retract A, Sets^n on any space A^n, a unital ring on $S^1 \times \mathbb{Z}$, a Boolean algebra on $\{0,1\}^{\aleph_0}$. Even, for any A and Σ, the Świerczkowski free algebra $\mathbf{F}_A(\Sigma)$ (based on a set larger than A).

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- E.g. the sphere $S^n \models \Sigma$ only for trivial Σ or for $n = 1, 3, 7$. (Hard algebraic topology to prove this.)
- There is no algorithm that settles $\mathbb{R} \models \Sigma$ for finite Σ. (Uses Matiasevich solution of Hilbert's Tenth Problem.) Thus $\models$ is not *too* sparse.

Relaxing the demands of $A \models \Sigma$.

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Approximate replacements for $A \models \Sigma$

For (A, d) a metric space, and $\eta > 0$

$$A \models_\eta \Sigma,$$

will mean that there exist *continuous* operations $\overline{F_t}$ on A satisfying Σ **within** η.

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(Of course we also study $A \models^\varepsilon_\eta \Sigma$!)

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- E.g. recursive enumerability of $A \models_\eta \Sigma$.

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Some further results

(δ, ε)-constraints.

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Given $\overline{F} \colon B \longrightarrow A$ (not necessarily continuous) and $\delta, \varepsilon > 0$, we say that $\overline{F}$ is (δ, ε)-**constrained** it satisfies: for all $b, b' \in B$, if $e(b, b') < \delta$, then $d(\overline{F}(b), \overline{F}(b')) < \varepsilon$.

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We say that $\overline{F}$ is *n*-constrained by (δ_0, δ_n) iff there exist $0 < \delta_0 \leq \delta_1 \leq \cdots \leq \delta_n$ such that $\overline{F}$ is (δ_0, δ_1)-constrained and (δ_1, δ_2)-constrained, and so on, up to (δ_{n-1}, δ_n)-constrained.

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(If $\overline{F}$ is uniformly continuous, then for every $\varepsilon > 0$ there exists $\delta > 0$ so that $\overline{F}$ is n-constrained by (δ, ε).)

The intermediate-value theorem, revisited.

Lemma

Suppose that f maps a convex subset of $\mathbb{R}$ into $\mathbb{R}$, and that f is (δ, ε)-constrained with $\delta, \varepsilon > 0$. If $a < c$ and s is between $f(a)$ and $f(c)$, then there exists b with $a \leq b \leq c$ and with $d(f(b), s) < \varepsilon/2$.

Definition of $\models_n^{\varepsilon}$ and $\mu_n(A, \Sigma)$

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means that *there exists an algebra* $\mathbf{A} = (A, \overline{F}_t)_{t \in T}$ *modeling* Σ *and a real number* $\delta_0 > 0$ *such that each* $\overline{F}_t$ *is* *n-constrained by* (δ_0, ε).

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It is not hard to see that

$$0 \ \leq \ \mu_1(A, \Sigma) \ \leq \ \mu_2(A, \Sigma) \ \leq \ \cdots \ \leq \ \mathrm{diam}(A).$$

Connection of $\models_n^\varepsilon$ and $\mu_n(A,\Sigma)$ with $A \models \Sigma$

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Thus if $A \models \Sigma$, then $A \models_n^\varepsilon \Sigma$ for every n and every $\varepsilon > 0$, and hence

$$\mu_n(A,\Sigma) \;=\; \inf\{\varepsilon : A \models_n^\varepsilon \Sigma\} \;=\; 0$$

for every n.

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"$\overline{G}$ is a one-one binary operation on A."

Σ will be this pair of equations:

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where $\overline{F}$ has $\overline{F}_0$ and $\overline{F}_1$ as its component functions. Thus $\overline{G}$ must be one-one and $\overline{F}$ must be onto.

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$$G'(a_0, b_0) \;=\; \varepsilon\,\overline{G}(a_0, b_0); \quad F'(a) \;=\; \overline{F}(1 \wedge (a/\varepsilon)).$$

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Now the discontinuities of G' are no larger than ε, and F' remains continuous, while F' and G' still satisfy Σ. Thus $A \models_1^\varepsilon$ for every $\varepsilon > 0$; hence $\mu_1([0,1],\Sigma) = 0$. ∎

$\mu_2([0,1],\Sigma) = 1$

We consider $(A; \overline{G}, \overline{F}_0, \overline{F}_1)$ modeling Σ, with the operations (δ_0, δ_1)-constrained and (δ_1, δ_2)-constrained. We will show that $\delta_2 \geq 1$.

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We consider the real function $\overline{H}(x) \;=\; \overline{G}(a_0, x)$. By the Lemma (IVT), there exists $e \in [0,1]$ with

$$d\left(\overline{G}(a_0, e), \overline{G}(a_1, b_0)\right)\Big) \;<\; \delta_1.$$

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Because $\{a_0, a_1\} = \{0, 1\}$, because of Σ, and because the function $\overline{F}_0$ is (δ_1, δ_2)-constrained, we now have:

$$1 \;=\; d(a_0, a_1) \;=\; d\big(\overline{F}_0(\overline{G}(a_0, e)), \overline{F}_0(\overline{G}(a_1, b_0))\big) \;\leq\; \delta_2.$$

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Thus $\mu_2([0,1],\Sigma)$, being the infimum of such δ_2's, must be 1.

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- $\mu_2([0,1]^n, \text{Groups}) \ = \ \text{diameter}([0,1]^n)$.
- $\mu_3(Y, \text{Lattices}) \ \geq \ 0.5$.

(Here Y stands for a Y-shaped one-dimensional space with each arm of unit length.)

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