Characterizing algebras and varieties by weak congruence lattices and open problems

Andreja Tepavčević and Branimir Šešelja
Department of Mathematics and Informatics, Faculty of Sciences
University of Novi Sad, Serbia

Blast 2010
Boulder, June 6. 2010.

Abstract

Abstract

Representation of an algebraic lattice by the weak congruence lattice of an algebra is still an open problem in universal algebra formulated 20 years ago. Its nontrivial version is to locate an element of a lattice representing the diagonal relation and then to find a corresponding algebra. There are solutions for some special cases, e.g., the diagonal being in the center of the lattice. Many sufficient conditions have also been obtained. The aim of the talk is to present the history of the topic and some recent new results.

Weak congruence relations

Weak congruence relations

A **weak congruence** on an algebra $\mathcal{A}$ is a symmetric and transitive subuniverse of $\mathcal{A}^2$.

Weak congruence relations

A **weak congruence** on an algebra $\mathcal{A}$ is a symmetric and transitive subuniverse of $\mathcal{A}^2$.

The weak congruences on $\mathcal{A}$ form an algebraic lattice under inclusion, denoted by $\mathrm{Con_w}(\mathcal{A})$.

Weak congruence relations

A **weak congruence** on an algebra $\mathcal{A}$ is a symmetric and transitive subuniverse of $\mathcal{A}^2$.

The weak congruences on $\mathcal{A}$ form an algebraic lattice under inclusion, denoted by $\mathrm{Con_w}(\mathcal{A})$.

The congruence lattice $\mathrm{Con}(\mathcal{A})$ of $\mathcal{A}$ is a principal filter in $\mathrm{Con_w}(\mathcal{A})$, generated by the diagonal relation Δ of A.

The congruence lattice of any subalgebra of $\mathcal{A}$ is an interval sublattice of $\mathrm{Con_w}(\mathcal{A})$.

The congruence lattice of any subalgebra of $\mathcal{A}$ is an interval sublattice of $\mathrm{Con_w}(\mathcal{A})$.

The subalgebra lattice $\mathrm{Sub}(\mathcal{A})$ *is isomorphic to the principal ideal generated by* Δ, *by sending each weak congruence* θ *contained in* Δ *to its domain*

$$A\theta = \{\, a \mid a\,\theta\, a\} = \{b \mid \exists\, a\ (a\,\theta\, b)\}.$$

The congruence lattice of any subalgebra of $\mathcal{A}$ is an interval sublattice of $\mathrm{Con_w}(\mathcal{A})$.

The subalgebra lattice $\mathrm{Sub}(\mathcal{A})$ *is isomorphic to the principal ideal generated by* Δ, *by sending each weak congruence* θ *contained in* Δ *to its domain*

$$A\theta = \{\, a \mid a\,\theta\, a\} = \{b \mid \exists\, a\ (a\,\theta\, b)\}.$$

Therefore, both the subalgebra lattice and the congruence lattice of an algebra may be recovered and investigated within a single algebraic lattice.

Theorem

(Chajda, Šešelja, Tepavčević, 1995) *A variety $\mathcal{V}$ which has a nullary operation in the similarity type is weak congruence modular if and only if $\mathcal{V}$ is polynomially equivalent to the variety of modules over a ring with unit.*

Theorem

(Chajda, Šešelja, Tepavčević, 1995) *A variety $\mathcal{V}$ which has a nullary operation in the similarity type is weak congruence modular if and only if $\mathcal{V}$ is polynomially equivalent to the variety of modules over a ring with unit.*

Theorem

(Šešelja, Tepavčević, 2001) *A congruence modular Abelian variety has the Congruence Intersection Property (the CIP) if and only if it has a constant term operation.*

Theorem

(Chajda, Šešelja, Tepavčević, 1995) *A variety $\mathcal{V}$ which has a nullary operation in the similarity type is weak congruence modular if and only if $\mathcal{V}$ is polynomially equivalent to the variety of modules over a ring with unit.*

Theorem

(Šešelja, Tepavčević, 2001) *A congruence modular Abelian variety has the Congruence Intersection Property (the CIP) if and only if it has a constant term operation.*

Open problem

Which (possibly locally finite) Abelian (or Hamiltonian) varieties possess the CIP?

Theorem

(Czédli, Šešelja, Tepavčević, 2007) *For any finite group G the following five conditions are equivalent.*

Theorem

(Czédli, Šešelja, Tepavčević, 2007) *For any finite group G the following five conditions are equivalent.*

- *G is a Dedekind group;*

Theorem

(Czédli, Šešelja, Tepavčević, 2007) *For any finite group G the following five conditions are equivalent.*

- *G is a Dedekind group;*
- *G has the CIP;*

Theorem

(Czédli, Šešelja, Tepavčević, 2007) *For any finite group G the following five conditions are equivalent.*

- *G is a Dedekind group;*
- *G has the CIP;*
- *Δ is a join-semidistributive element in the weak congruence lattice of G;*

Theorem

(Czédli, Šešelja, Tepavčević, 2007) *For any finite group G the following five conditions are equivalent.*

- *G is a Dedekind group;*
- *G has the CIP;*
- *Δ is a join-semidistributive element in the weak congruence lattice of G;*
- *for every normal subgroup N of G,*

$$C_N := \{K \in Sub(G) : \exists H \in Nor(K) \text{ with } (H)_G = N\}$$

is a sublattice of $Sub(G)$;

Theorem

(Czédli, Šešelja, Tepavčević, 2007) *For any finite group G the following five conditions are equivalent.*

- *G is a Dedekind group;*
- *G has the CIP;*
- *Δ is a join-semidistributive element in the weak congruence lattice of G;*
- *for every normal subgroup N of G,*

$$C_N := \{K \in Sub(G) : \exists H \in Nor(K) \text{ with } (H)_G = N\}$$

  is a sublattice of $Sub(G)$;
- *for every normal subgroup N of G, C_N is closed with respect to intersection.*

Theorem

(Czédli, Erné, Šešelja, Tepavčević, 2010) *A group is a Dedekind group if and only if its weak congruence lattice is modular.*

Theorem

(Czédli, Erné, Šešelja, Tepavčević, 2010) *The following statements on a group G are equivalent:*

Theorem

(Czédli, Erné, Šešelja, Tepavčević, 2010) *The following statements on a group G are equivalent:*
(*i*) *G is a Dedekind group.*

Theorem

(Czédli, Erné, Šešelja, Tepavčević, 2010) *The following statements on a group G are equivalent:*
(*i*) *G is a Dedekind group.*
(*ii*) $\mathrm{Con_w}(G)$ *is modular.*

Theorem

(Czédli, Erné, Šešelja, Tepavčević, 2010) *The following statements on a group G are equivalent:*
(*i*) *G is a Dedekind group.*
(*ii*) $\mathrm{Con_w}(G)$ *is modular.*
(*iii*) Δ *is a standard (equivalently, a neutral) element of* $\mathrm{Con_w}(G)$.

Theorem

(Czédli, Erné, Šešelja, Tepavčević, 2010) *The following statements on a group G are equivalent:*
(i) *G is a Dedekind group.*
(ii) $\mathrm{Con_w}(G)$ *is modular.*
(iii) Δ *is a standard (equivalently, a neutral) element of* $\mathrm{Con_w}(G)$.
(iv) *G has the* CIP *and the* CEP.

Theorem

(Czédli, Erné, Šešelja, Tepavčević, 2010) *The following statements on a group G are equivalent:*
(i) *G is a Dedekind group.*
(ii) $\mathrm{Con_w}(G)$ *is modular.*
(iii) Δ *is a standard (equivalently, a neutral) element of* $\mathrm{Con_w}(G)$.
(iv) *G has the* CIP *and the* CEP.

Corollary

(Czédli, Erné, Šešelja, Tepavčević, 2010) *A group is locally cyclic if and only if its weak congruence lattice is distributive.*

We call an algebra $\mathcal{A}$ **group-like** if it has a least subuniverse $\{e\}$ and there is some function $q : A^2 \to A$ such that for all $\theta \in \mathrm{Con_w}(\mathcal{A})$,

$$a\,\theta\,b \;\Leftrightarrow\; e\,\theta\,q(a,b) \text{ and } a,b \in A\theta\ .$$

We call an algebra $\mathcal{A}$ **group-like** if it has a least subuniverse $\{e\}$ and there is some function $q : A^2 \to A$ such that for all $\theta \in \mathrm{Con_w}(\mathcal{A})$,

$$a\,\theta\,b \;\Leftrightarrow\; e\,\theta\,q(a,b) \text{ and } a,b \in A\theta\ .$$

A group-like algebra $\mathcal{A}$ a **Dedekind algebra** if every subalgebra of $\mathcal{A}$ is a kernel, i.e., of the form $e\theta$ for some $\theta \in \mathrm{Con}(A)$.

Theorem

(Czédli, Erné, Šešelja, Tepavčević, 2010) *Let $\mathcal{A}$ be a group-like algebra that is a join of Dedekind subalgebras. Then the following statements are equivalent:*

Theorem

(Czédli, Erné, Šešelja, Tepavčević, 2010) *Let $\mathcal{A}$ be a group-like algebra that is a join of Dedekind subalgebras. Then the following statements are equivalent:*

- $\mathcal{A}$ *is a Dedekind algebra.*

Theorem

(Czédli, Erné, Šešelja, Tepavčević, 2010) *Let $\mathcal{A}$ be a group-like algebra that is a join of Dedekind subalgebras. Then the following statements are equivalent:*

- *$\mathcal{A}$ is a Dedekind algebra.*
- $\mathrm{Con_w}(\mathcal{A})$ *admits an isomorphism onto* $\mathrm{Sub}(\mathcal{A})_{\geq}$ *sending* Δ *to* $(A, \{e\})$.

Theorem

(Czédli, Erné, Šešelja, Tepavčević, 2010) *Let $\mathcal{A}$ be a group-like algebra that is a join of Dedekind subalgebras. Then the following statements are equivalent:*

- *$\mathcal{A}$ is a Dedekind algebra.*
- $\mathrm{Con_w}(\mathcal{A})$ *admits an isomorphism onto* $\mathrm{Sub}(\mathcal{A})_{\geq}$ *sending* Δ *to* $(A, \{e\})$.
- $\mathrm{Con_w}(\mathcal{A})$ *admits a lattice embedding in* $\mathrm{Sub}(\mathcal{A})^2$ *sending* Δ *to* $(A, \{e\})$.

Theorem

(Czédli, Erné, Šešelja, Tepavčević, 2010) *Let $\mathcal{A}$ be a group-like algebra that is a join of Dedekind subalgebras. Then the following statements are equivalent:*

- *$\mathcal{A}$ is a Dedekind algebra.*
- $\mathrm{Con_w}(\mathcal{A})$ *admits an isomorphism onto* $\mathrm{Sub}(\mathcal{A})_{\geq}$ *sending* Δ *to* $(A, \{e\})$.
- $\mathrm{Con_w}(\mathcal{A})$ *admits a lattice embedding in* $\mathrm{Sub}(\mathcal{A})^2$ *sending* Δ *to* $(A, \{e\})$.
- Δ *is a standard (equivalently, a neutral) element of* $\mathrm{Con_w}(\mathcal{A})$.

Theorem

(Czédli, Erné, Šešelja, Tepavčević, 2010) *Let $\mathcal{A}$ be a group-like algebra that is a join of Dedekind subalgebras. Then the following statements are equivalent:*

- *$\mathcal{A}$ is a Dedekind algebra.*
- $\mathrm{Con_w}(\mathcal{A})$ *admits an isomorphism onto* $\mathrm{Sub}(\mathcal{A})_{\geq}$ *sending* Δ *to* $(A, \{e\})$.
- $\mathrm{Con_w}(\mathcal{A})$ *admits a lattice embedding in* $\mathrm{Sub}(\mathcal{A})^2$ *sending* Δ *to* $(A, \{e\})$.
- Δ *is a standard (equivalently, a neutral) element of* $\mathrm{Con_w}(\mathcal{A})$.
- $\mathcal{A}$ *has the* CIP *and the* CEP.

Theorem

(Czédli, Erné, Šešelja, Tepavčević, 2010) *Let $\mathcal{A}$ be a group-like algebra that is a join of Dedekind subalgebras. Then the following statements are equivalent:*

- *$\mathcal{A}$ is a Dedekind algebra.*
- $\mathrm{Con_w}(\mathcal{A})$ *admits an isomorphism onto* $\mathrm{Sub}(\mathcal{A})_{\geq}$ *sending* Δ *to* $(A, \{e\})$.
- $\mathrm{Con_w}(\mathcal{A})$ *admits a lattice embedding in* $\mathrm{Sub}(\mathcal{A})^2$ *sending* Δ *to* $(A, \{e\})$.
- *Δ is a standard (equivalently, a neutral) element of* $\mathrm{Con_w}(\mathcal{A})$.
- *$\mathcal{A}$ has the* CIP *and the* CEP.

Moreover, the weak congruence lattice $\mathrm{Con_w}(\mathcal{A})$ *is modular (distributive) if and only if $\mathcal{A}$ is a Dedekind algebra with modular (distributive) subalgebra lattice* $\mathrm{Sub}(\mathcal{A})$.

Corollary

(Czédli, Erné, Šešelja, Tepavčević, 2010) *A ring is Hamiltonian if and only if it is generated by Hamiltonian subrings and has a modular weak congruence lattice or* Δ *is a neutral element of it.*

Representation of lattices by weak congruences

Representation of lattices by weak congruences

Bacic representation problem

Represent an algebraic lattice by the weak congruence lattice of an algebra.

Representation of lattices by weak congruences

Bacic representation problem

Represent an algebraic lattice by the weak congruence lattice of an algebra.

Easily solved by Grätzer-Schmidt theorem:
Let $\mathcal{B} = (A, F)$ be an algebra such that $\operatorname{Con}\mathcal{B}$ is isomorphic with L. Then the required algebra $\mathcal{A}$ can be obtained by adding to F all the elements from A as nullary operations: $\mathcal{A} = (A, F \cup \{A\})$.
Obviously, $\operatorname{Con}_{\mathrm{w}}(\mathcal{A}) \cong \operatorname{Con}\mathcal{B} \cong L$.

Representation of lattices by weak congruences

Bacic representation problem

Represent an algebraic lattice by the weak congruence lattice of an algebra.

Easily solved by Grätzer-Schmidt theorem:
Let $\mathcal{B} = (A, F)$ be an algebra such that $\operatorname{Con}\mathcal{B}$ is isomorphic with L. Then the required algebra $\mathcal{A}$ can be obtained by adding to F all the elements from A as nullary operations: $\mathcal{A} = (A, F \cup \{A\})$.
Obviously, $\operatorname{Con}_{\mathrm{w}}(\mathcal{A}) \cong \operatorname{Con}\mathcal{B} \cong L$.

The above construction by which the diagonal relation of the algebra corresponds to the bottom of the lattice is called the **trivial representation**.

Weak congruence lattice representation problem 1

Let L be an algebraic lattice and $a \in L$. Find an algebra such that its weak congruence lattice is isomorphic with L, the diagonal relation being the image of a under the isomorphism.

Weak congruence lattice representation problem 1

Let L be an algebraic lattice and $a \in L$. Find an algebra such that its weak congruence lattice is isomorphic with L, the diagonal relation being the image of a under the isomorphism.

A representation by which the diagonal relation corresponds to an element different from the bottom of the lattice is said to be **non-trivial**.

Weak congruence lattice representation problem 1

Let L be an algebraic lattice and $a \in L$. Find an algebra such that its weak congruence lattice is isomorphic with L, the diagonal relation being the image of a under the isomorphism.

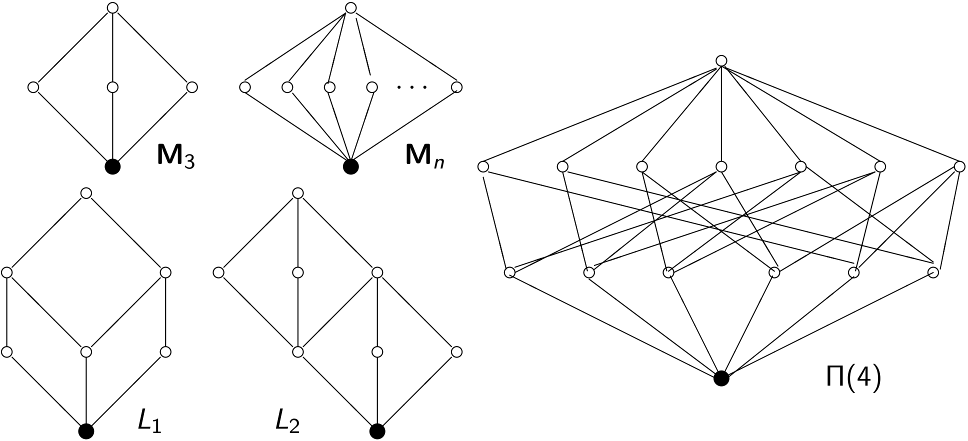
A representation by which the diagonal relation corresponds to an element different from the bottom of the lattice is said to be **non-trivial**.

Weak congruence lattice representation problem 2

Find a non-trivial representation of an algebraic lattice by a weak congruence lattice of an algebra.

Examples: lattices without non-trivial representations

Examples: lattices without non-trivial representations



Δ-suitable elements of a lattice

Δ-suitable elements of a lattice

Let L be an algebraic lattice. An element $a \in L$ is said to be **Δ-suitable** if there is an algebra $\mathcal{A}$ such that the weak congruence lattice $\mathrm{Con_w}(\mathcal{A})$ is isomorphic to L, and Δ corresponds to a under the isomorphism.

Δ-suitable elements of a lattice

Let L be an algebraic lattice. An element $a \in L$ is said to be **Δ-suitable** if there is an algebra $\mathcal{A}$ such that the weak congruence lattice $\mathrm{Con_w}(\mathcal{A})$ is isomorphic to L, and Δ corresponds to a under the isomorphism.

Proposition

Every Δ-suitable element of a lattice is co-distributive.

If a is a co-distributive element of L, then the mapping $\mu : x \mapsto x \wedge a$ is an endomorphism on L.

If a is a co-distributive element of L, then the mapping $\mu : x \mapsto x \wedge a$ is an endomorphism on L.

Notation:

If a is a co-distributive element of L, then the mapping $\mu : x \mapsto x \wedge a$ is an endomorphism on L.

Notation:

θ_a - a congruence on L which is the kernel of μ;

If a is a co-distributive element of L, then the mapping $\mu : x \mapsto x \wedge a$ is an endomorphism on L.

Notation:

θ_a - a congruence on L which is the kernel of μ;
$\overline{x}$ - the greatest element of the block $[x]_{\theta_a}$, $x \in L$;

If a is a co-distributive element of L, then the mapping $\mu : x \mapsto x \wedge a$ is an endomorphism on L.

Notation:

θ_a - a congruence on L which is the kernel of μ;
$\overline{x}$ - the greatest element of the block $[x]_{\theta_a}$, $x \in L$;
$M_a := \{\overline{x} \mid x \in L\}$.

Proposition

A Δ-suitable element $a \in L$ satisfies the following:

Proposition

A Δ-suitable element $a \in L$ satisfies the following:

- *if* $x \wedge y \neq \mathbf{0}$ *then* $\overline{x \vee y} = \overline{x} \vee \overline{y}$;

Proposition

A Δ-suitable element $a \in L$ satisfies the following:

- *if* $x \wedge y \neq \mathbf{0}$ *then* $\overline{x \vee y} = \overline{x} \vee \overline{y}$;
- *if* $x \neq \mathbf{0}$ *and* $\overline{x} < y$, *then* $\overline{y \wedge a} \neq y \wedge a$;

Proposition

A Δ-suitable element $a \in L$ satisfies the following:

- *if* $x \wedge y \neq \mathbf{0}$ *then* $\overline{x \vee y} = \overline{x} \vee \overline{y}$;
- *if* $x \neq \mathbf{0}$ *and* $\overline{x} < y$, *then* $\overline{y \wedge a} \neq y \wedge a$;
- *if* $x \prec a$, *then* $\bigvee(y \in \uparrow a \mid y \vee \overline{x} < \mathbf{1}) \neq \mathbf{1}$;

Proposition

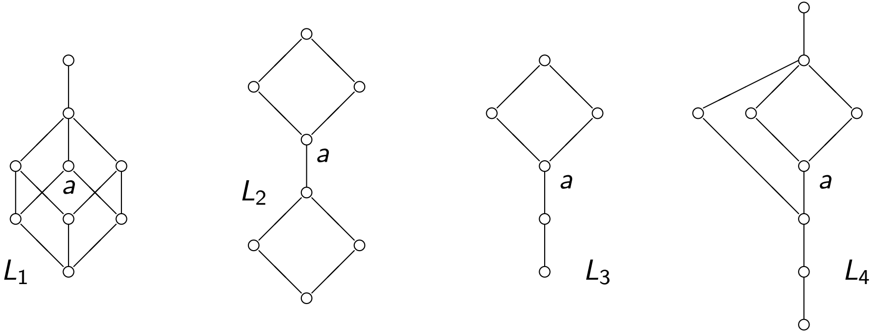
A Δ-suitable element $a \in L$ satisfies the following:

- *if* $x \wedge y \neq \mathbf{0}$ *then* $\overline{x \vee y} = \overline{x} \vee \overline{y}$;
- *if* $x \neq \mathbf{0}$ *and* $\overline{x} < y$, *then* $\overline{y \wedge a} \neq y \wedge a$;
- *if* $x \prec a$, *then* $\bigvee(y \in \uparrow a \mid y \vee \overline{x} < \mathbf{1}) \neq \mathbf{1}$;
- *If* $y \in \downarrow a$ *and* $x \prec y$, *then there exists* $z \in [y, \overline{y}]$, *such that*
  - *for all* $t \in [x, \overline{x}]$, *the set* $\{c \in \mathrm{Ext}(t) \mid c \leqslant z\}$ *is either empty or has the top element, and*
  - *for all* $t \in [x, \overline{x}]$, *the set* $\{c \in \mathrm{Ext}(t) \mid c \nleqslant z\}$ *is an antichain (possibly empty), where*

$$\mathrm{Ext}(t) := \{w \in [y, \overline{y}] \mid w \cap \overline{x} = t\}.$$

Examples

Examples



Proposition

If a is a Δ-suitable element of a lattice L and

$$|[\mathbf{0}]_{\theta_a}| > 1 \text{ or there is a single atom in } L \,,$$

then M_a is a sublattice of L.

Proposition

If a is a Δ-suitable element of a lattice L and

$$|[\mathbf{0}]_{\theta_a}| > 1 \text{ or there is a single atom in } L \, ,$$

then M_a is a sublattice of L.

Proposition

In a lattice with more than 2 elements, the top element is not Δ-suitable.

Proposition

If a is a Δ-suitable element of the lattice L, then the following hold:

Proposition

If a is a Δ-suitable element of the lattice L, then the following hold:

- $x \wedge a < y \wedge a$ *implies* $\overline{x} \vee a < \overline{y} \vee a$ *for all* $x, y \in L$ *if and only if every algebra representing* L *is Hamiltonian;*

Proposition

If a is a Δ-suitable element of the lattice L, then the following hold:

- $x \wedge a < y \wedge a$ *implies* $\overline{x} \vee a < \overline{y} \vee a$ *for all* $x, y \in L$ *if and only if every algebra representing* L *is Hamiltonian;*
- a *is a cancellable element in* L *if and only if every algebra representing* L *has the CEP;*

Proposition

If a is a Δ-suitable element of the lattice L, then the following hold:

- $x \wedge a < y \wedge a$ *implies* $\overline{x} \vee a < \overline{y} \vee a$ *for all* $x, y \in L$ *if and only if every algebra representing* L *is Hamiltonian;*
- a *is a cancellable element in* L *if and only if every algebra representing* L *has the CEP;*
- a *is a distributive element in* L *if and only if every algebra representing* L *has the CIP;*

Proposition

If a is a Δ-suitable element of the lattice L, then the following hold:

- $x \wedge a < y \wedge a$ *implies* $\overline{x} \vee a < \overline{y} \vee a$ *for all* $x, y \in L$ *if and only if every algebra representing* L *is Hamiltonian;*
- a *is a cancellable element in* L *if and only if every algebra representing* L *has the CEP;*
- a *is a distributive element in* L *if and only if every algebra representing* L *has the CIP;*
- $\overline{x} \vee a = \mathbf{1}$ *for every* $x \in L$ *if and only if no congruence on an algebra representing* L *has a block which is a proper subalgebra;*

- *$x \prec a$ implies $\overline{x} \vee a < 1$ for every $x \in L$ if and only if every algebra representing L is quasi-Hamiltonian;*

- *$x \prec a$ implies $\overline{x} \vee a < 1$ for every $x \in L$ if and only if every algebra representing L is quasi-Hamiltonian;*
- *a has a complement in L if and only if every algebra representing L has at least one nullary operation and has no congruence whose block is a proper subalgebra.*

Theorem

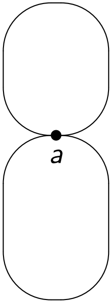
Let $L = \downarrow a \cup \uparrow a$, $a \in L$. *If* a *is* Δ*-suitable, then:*

- $\downarrow a$ *is a two-element chain, or* $\mathbf{M}_n$ *for some* $n \in \mathbb{N}$ *or* $\mathbf{M}_\infty$;

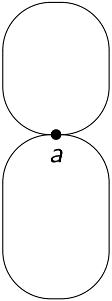
Theorem

Let $L = \downarrow a \cup \uparrow a$, $a \in L$. If a is Δ-suitable, then:

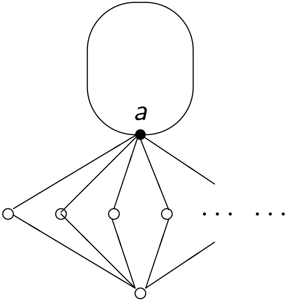
- $\downarrow a$ *is a two-element chain, or* $\mathbf{M}_n$ *for some* $n \in \mathbb{N}$ *or* $\mathbf{M}_\infty$;
- *any algebra representing L has at most one-element subalgebras, it satisfies the CEP and the CIP, it is Hamiltonian, and if $\downarrow a$ is not a two-element chain then it has no constants.*



L



L



L

Theorem

If a is a Δ-suitable element belonging to the center of a lattice L, then every algebra representing L satisfies the following:

Theorem

If a is a Δ-suitable element belonging to the center of a lattice L, then every algebra representing L satisfies the following:

- *$\mathcal{A}$ has at least one nullary operation;*

Theorem

If a is a Δ-suitable element belonging to the center of a lattice L, then every algebra representing L satisfies the following:

- *$\mathcal{A}$ has at least one nullary operation;*
- *$\mathcal{A}$ has the CEP and the CIP;*

Theorem

If a is a Δ-suitable element belonging to the center of a lattice L, then every algebra representing L satisfies the following:

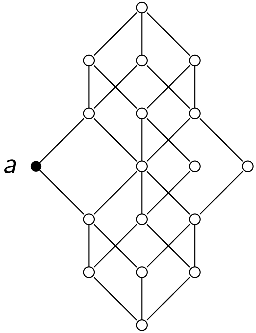
- *$\mathcal{A}$ has at least one nullary operation;*
- *$\mathcal{A}$ has the CEP and the CIP;*
- *for every subalgebra $\mathcal{B}$ of $\mathcal{A}$, $\mathrm{Con}\,\mathcal{B}$ is isomorphic with $\mathrm{Con}\,\mathcal{A}$;*

Theorem

If a is a Δ-suitable element belonging to the center of a lattice L, then every algebra representing L satisfies the following:

- *$\mathcal{A}$ has at least one nullary operation;*
- *$\mathcal{A}$ has the CEP and the CIP;*
- *for every subalgebra $\mathcal{B}$ of $\mathcal{A}$, $\mathrm{Con}\,\mathcal{B}$ is isomorphic with $\mathrm{Con}\,\mathcal{A}$;*
- *$\mathcal{A}$ is not Hamiltonian, moreover no congruence on $\mathcal{A}$ has a block which is a subalgebra of $\mathcal{A}$.*

Example



In the free distributive lattice with three generators, the generating elements (and, trivially, the bottom) are the only ones which are Δ*-suitable.*

In the free distributive lattice with three generators, the generating elements (and, trivially, the bottom) are the only ones which are Δ-suitable.
Hence, there is one possible non-trivial representation of this lattice.

In the free distributive lattice with three generators, the generating elements (and, trivially, the bottom) are the only ones which are Δ-suitable.
Hence, there is one possible non-trivial representation of this lattice.
Every algebra representing this lattice:

In the free distributive lattice with three generators, the generating elements (and, trivially, the bottom) are the only ones which are Δ*-suitable.*
Hence, there is one possible non-trivial representation of this lattice.
Every algebra representing this lattice:

- *is non-Hamiltonian;*

In the free distributive lattice with three generators, the generating elements (and, trivially, the bottom) are the only ones which are Δ*-suitable.*
Hence, there is one possible non-trivial representation of this lattice.
Every algebra representing this lattice:

- *is non-Hamiltonian;*
- *has nullary operations;*

In the free distributive lattice with three generators, the generating elements (and, trivially, the bottom) are the only ones which are Δ-suitable.
Hence, there is one possible non-trivial representation of this lattice.
Every algebra representing this lattice:

- *is non-Hamiltonian;*
- *has nullary operations;*
- *satisfy the CEP and the CIP;*

In the free distributive lattice with three generators, the generating elements (and, trivially, the bottom) are the only ones which are Δ-suitable.
Hence, there is one possible non-trivial representation of this lattice.
Every algebra representing this lattice:

- *is non-Hamiltonian;*
- *has nullary operations;*
- *satisfy the CEP and the CIP;*
- *all its proper subalgebras are Hamiltonian.*

In a complete and atomic Boolean algebra $\mathcal{B}$ all elements except the top are Δ-suitable.

In a complete and atomic Boolean algebra $\mathcal{B}$ all elements except the top are Δ-suitable.
Hence, for every $a \in B$, $a \neq \mathbf{1}$, there could be a representation of $\mathcal{B}$ by weak congruences.

In a complete and atomic Boolean algebra $\mathcal{B}$ all elements except the top are Δ-suitable.
Hence, for every $a \in B$, $a \neq \mathbf{1}$, there could be a representation of $\mathcal{B}$ by weak congruences.
The representation is not trivial if $a \neq \mathbf{0}$. In this case, every algebra $\mathcal{A}$ representing $\mathcal{B}$ has the following properties:

In a complete and atomic Boolean algebra $\mathcal{B}$ all elements except the top are Δ-suitable.
Hence, for every $a \in B$, $a \neq \mathbf{1}$, there could be a representation of $\mathcal{B}$ by weak congruences.
The representation is not trivial if $a \neq \mathbf{0}$. In this case, every algebra $\mathcal{A}$ representing $\mathcal{B}$ has the following properties:

- *$\mathcal{A}$ has at least one nullary operation in its similarity type;*

In a complete and atomic Boolean algebra $\mathcal{B}$ all elements except the top are Δ-suitable.
Hence, for every $a \in B$, $a \neq \mathbf{1}$, there could be a representation of $\mathcal{B}$ by weak congruences.
The representation is not trivial if $a \neq \mathbf{0}$. In this case, every algebra $\mathcal{A}$ representing $\mathcal{B}$ has the following properties:

- *$\mathcal{A}$ has at least one nullary operation in its similarity type;*
- *$\mathcal{A}$ satisfies the CEP and the CIP;*

In a complete and atomic Boolean algebra $\mathcal{B}$ all elements except the top are Δ-suitable.
Hence, for every $a \in B$, $a \neq \mathbf{1}$, there could be a representation of $\mathcal{B}$ by weak congruences.
The representation is not trivial if $a \neq \mathbf{0}$. In this case, every algebra $\mathcal{A}$ representing $\mathcal{B}$ has the following properties:

- *$\mathcal{A}$ has at least one nullary operation in its similarity type;*
- *$\mathcal{A}$ satisfies the CEP and the CIP;*
- *$\mathcal{A}$ is not Hamiltonian, neither quasi-Hamiltonian;*

In a complete and atomic Boolean algebra $\mathcal{B}$ all elements except the top are Δ-suitable.
Hence, for every $a \in B$, $a \neq \mathbf{1}$, there could be a representation of $\mathcal{B}$ by weak congruences.
The representation is not trivial if $a \neq \mathbf{0}$. In this case, every algebra $\mathcal{A}$ representing $\mathcal{B}$ has the following properties:

- *$\mathcal{A}$ has at least one nullary operation in its similarity type;*
- *$\mathcal{A}$ satisfies the CEP and the CIP;*
- *$\mathcal{A}$ is not Hamiltonian, neither quasi-Hamiltonian;*
- *no congruence on $\mathcal{A}$ has a block which is a subalgebra of $\mathcal{A}$;*

In a complete and atomic Boolean algebra $\mathcal{B}$ all elements except the top are Δ-suitable.
Hence, for every $a \in B$, $a \neq \mathbf{1}$, there could be a representation of $\mathcal{B}$ by weak congruences.
The representation is not trivial if $a \neq \mathbf{0}$. In this case, every algebra $\mathcal{A}$ representing $\mathcal{B}$ has the following properties:

- *$\mathcal{A}$ has at least one nullary operation in its similarity type;*
- *$\mathcal{A}$ satisfies the CEP and the CIP;*
- *$\mathcal{A}$ is not Hamiltonian, neither quasi-Hamiltonian;*
- *no congruence on $\mathcal{A}$ has a block which is a subalgebra of $\mathcal{A}$;*
- *all congruence lattices of subalgebras of $\mathcal{A}$ are isomorphic with $\mathrm{Con}\,\mathcal{A}$.*

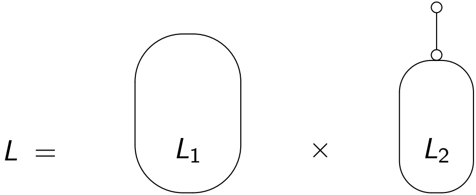
Particular solution

Particular solution

A nontrivial representation problem is solved in a case $L = L_1 \times L_2$ where L_1 and L_2 are algebraic lattices, L_2 having a single co-atom.

Particular solution

A nontrivial representation problem is solved in a case $L = L_1 \times L_2$ where L_1 and L_2 are algebraic lattices, L_2 having a single co-atom.



Theorem

Let L be an algebraic lattice and $a \in L$ an element from the center of the lattice, such that $\uparrow a$ has a single co-atom.

Theorem

Let L be an algebraic lattice and $a \in L$ an element from the center of the lattice, such that $\uparrow a$ has a single co-atom.
Then, there is an algebra $\mathcal{A}$, whose weak congruence lattice $\mathrm{Con_w}(\mathcal{A})$ is isomorphic with L under a mapping sending Δ to a.

Open problem

Find conditions under which $\mathrm{Con_w}(\mathcal{A})$ *is isomorphic with* $\mathrm{Con}(\mathcal{B})$ *for an algebra* $\mathcal{B}$ *and the following holds:*
$\mathcal{A} \cong \mathcal{B}/\theta$ *for some congruence* θ *on* $\mathcal{B}$*.*

References

References

- G. Vojvodić, B. Šešelja, *On the lattice of weak congruence relations*, Algebra Univers. **25** (1988) 121-130.
- A. Tepavčević, *On representation of lattices by weak congruences and weak tolerances*, Algebra and Model Theory, ed. by A.G. Pinus and K.N. Ponomaryov, Novosibirsk, 1997, 173-181.
- I. Chajda, B. Šešelja, A. Tepavčević, *Lattices of compatible relations satisfying a set of formulas*, Algebra Univers. **40** (1998) 51-58.
- B. Šešelja, A. Tepavčević, *Relative complements in the lattice of weak congruences*, Publ. Inst. Math. Beograd **67** (81) (2000) 7-13.

- B. Šešelja, A. Tepavčević, *Weak Congruences in Universal Algebra*, Institute of Mathematics Novi Sad, 2001.
- B. Šešelja, A. Tepavčević, *A note on CIP varieties*, Algebra Univers. **45** (2001) 349-351.
- G. Czedli, B. Šešelja, A. Tepavčević, *Semidistributive elements in lattices; application to groups and rings*, Algebra Univers. **58** (2008) 349-355.
- G. Czedli, M. Erné, B. Šešelja, A. Tepavčević, *Characteristic triangles of closure operators with applications in general algebra*, Algebra Univers. (to appear).

Thank you for your attention!