Tutorial on Universal Algebra, Mal'cev Conditions, and Finite Relational Structures: Lecture I

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BLAST 2010

Boulder, June 2010

Outline - Lecture 1

0. Apology

Part I: Basic universal algebra

1. Algebras, terms, identities, varieties
2. Interpretations of varieties
3. The lattice $\mathcal{L}$, filters, Mal'cev conditions

Part II: Duality between finite algebras and finite relational structures

4. Relational structures and the pp-interpretability ordering
5. Polymorphisms and the connection to algebra

Outline (continued) – Lecture 2

PART III: The Constraint Satisfaction Problem

6. The CSP dichotomy conjecture of Feder and Vardi
7. Connections to $(\mathcal{R}_{\text{fin}}, \leq_{\text{pp}})$ and Mal'cev conditions
8. New Mal'cev conditions (Maróti, McKenzie; Barto, Kozik)
9. New proof of an old theorem of Hell-Nešetřil via algebra (Barto, Kozik)
10. Current status, open problems.

0. Apology

I'm sorry

Part I. Basic universal algebra

algebra: a structure $\mathbf{A} = (A; \{\textit{fundamental operations}\})$[1]

term: expression $t(\mathbf{x})$ built from fundamental operations and variables.

- term t in n variables defines an n-ary **term operation** $t^{\mathbf{A}}$ on A.

Definition

$$\mathrm{TermOps}(\mathbf{A}) = \{t^{\mathbf{A}} \ : \ t \text{ a term in } n \geq 1 \text{ variables}\}.$$

Definition

$\mathbf{A}, \mathbf{B}$ are **term-equivalent** if they have the same universe and same term operations.

[1] Added post-lecture: For these notes, algebras are *not* permitted nullary operations

identity: first-order sentence of the form $\forall \mathbf{x}(s = t)$ with s, t terms.

- Notation: $s \approx t$.

Definition

A **variety** (or **equational class**) is any class of algebras (in a fixed language) axiomatizable by identities.

Examples:

- {semigroups}; {groups} (in language $\{\cdot, {}^{-1}\}$).
- $\mathrm{var}(\mathbf{A}) :=$ variety axiomatized by all identities true in $\mathbf{A}$.

Definition

Say varieties V, W are **term-equivalent**, and write $V \equiv W$, if:

- Every $\mathbf{A} \in V$ is term-equivalent to some $\mathbf{B} \in W$ and vice versa ...
- ... "uniformly and mutually inversely."

Example: {boolean algebras} $\equiv$ {idempotent ($x^2 \approx x$) rings}.

Definition

Given an algebra $\mathbf{A} = (A; F)$ and a subset $S \subseteq \mathrm{TermOps}(\mathbf{A})$, the algebra $(A; S)$ is a **term reduct** of $\mathbf{A}$.

Definition

Given varieties V, W, write $V \rightarrow W$ and say that V is **interpretable** in W if every member of $\underline{W}$ has a term reduct belonging to $\underline{V}$.

Examples:

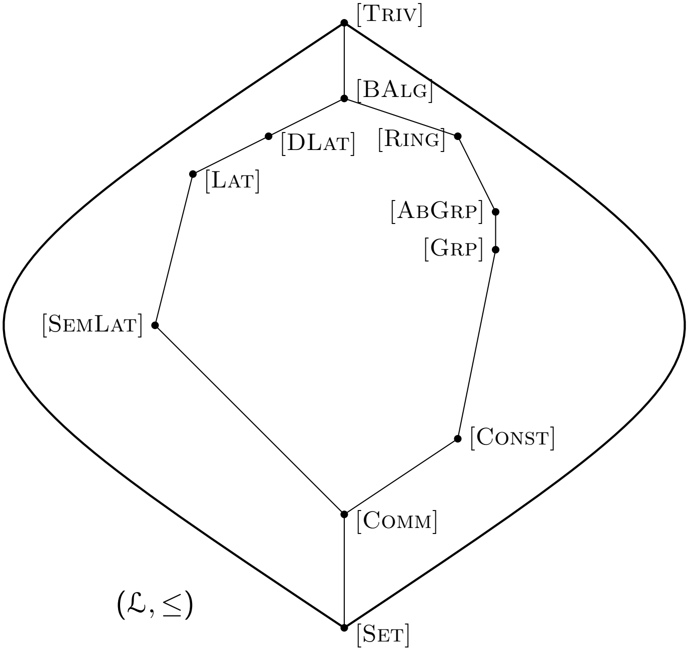
$$\begin{array}{rcll} \textsc{Groups} & \rightarrow & \textsc{Rings}, & \text{but} \quad \textsc{Rings} \not\rightarrow \textsc{Groups} \\ \textsc{Groups} & \rightarrow & \textsc{AbelGrps} & \\ \text{More generally,} \quad V & \rightarrow & W & \text{whenever } W \subseteq V \\ \textsc{Sets} & \rightarrow & V & \text{for any variety } V \\ \textsc{Semigrps} & \rightarrow & \textsc{Sets} & \end{array}$$

Intuition: $V \rightarrow W$ if it is "at least as hard" to construct a nontrivial member of W as it is for V. ("Nontrivial" = universe has ≥ 2 elements.)

The relation $\to$ on varieties is a pre-order (reflexive and transitive).

So we get a partial order in the usual way:

$$\begin{aligned} V \sim W \quad & \text{iff} \quad V \to W \to V \\ [V] \quad & = \quad \{W : V \sim W\} \\ \mathcal{L} \quad & = \quad \{[V] : V \text{ a variety}\} \\ [V] \leq [W] \quad & \text{iff} \quad V \to W. \end{aligned}$$


[TRIV]
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[ABGRP]
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[CONST]
[COMM]
[SET]
$(\mathcal{L}, \leq)$

Remarks:

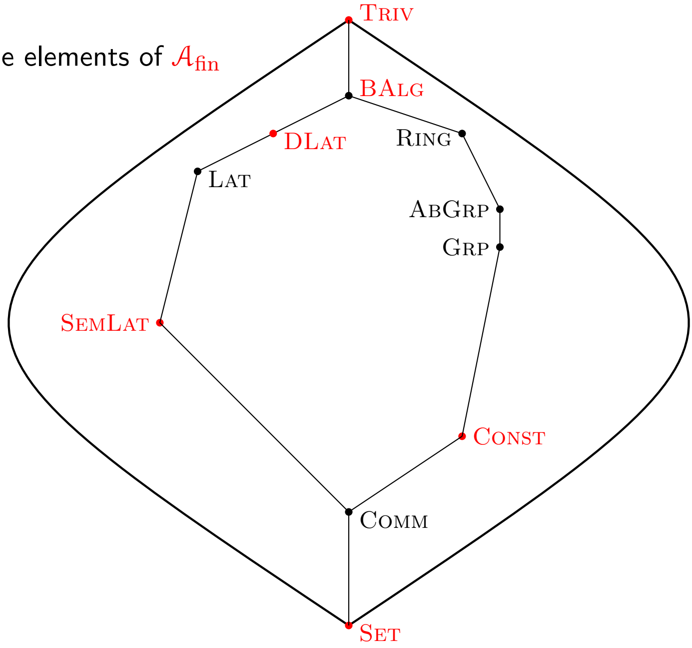
- $(\mathcal{L}, \leq)$ defined by W.D. Neumann (1974); studied by Garcia, Taylor (1984).
- $\mathcal{L}$ is a proper class.
- $(\mathcal{L}, \leq)$ is a complete lattice.
- $\mathcal{L}_\kappa := \{[V]$: the language of V has card $\leq \kappa\}$ is a set and a sublattice of $\mathcal{L}$.

Also note: every algebra **A** "appears" in $\mathcal{L}$, i.e. as $[\mathrm{var}(\mathbf{A})]$.

Of particular interest: $\mathcal{A}_{\mathrm{fin}} := \{[\mathrm{var}(\mathbf{A})]$: **A** a finite algebra$\}$.

- $\mathcal{A}_{\mathrm{fin}}$ is a countable $\wedge$-closed sub-poset of $\mathcal{L}_\omega$.

Some elements of $\mathcal{A}_{\text{fin}}$



Thesis: "good" classes of varieties invariably form **filters** in $\mathcal{L}$ of a special kind: they are generated by a set of **finitely presented varieties**[2].

Definition

Such a filter in $\mathcal{L}$ (or the class of varieties represented in the filter) is called a **Mal'cev class** (or **condition**).

Bad example of a Mal'cev class: the class $\mathcal{C}$ of varieties V which, for some n, have a $2n$-ary term $t(x_1, \ldots, x_{2n})$ satisfying

$$V \models t(x_1, x_2, \ldots, x_{2n}) \approx t(x_{2n}, \ldots, x_2, x_1).$$

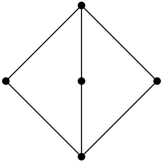
If we let U_n have a single $2n$-ary operation f and a single axiom $f(x_1, \ldots, x_{2n}) \approx f(x_{2n}, \ldots, x_1)$, then $\mathcal{C}$ corresponds to the filter in $\mathcal{L}$ generated by $\{[U_n] : n \geq 1\}$.

[2] finite language and axiomatized by finitely many identities

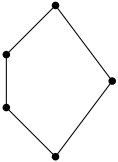
Better example: congruence modularity

Every algebra **A** has an associated lattice $\mathrm{Con}(\mathbf{A})$, called its **congruence lattice**, analogous to the lattice of normal subgroups of a group, or the lattice of ideals of a ring.

The **modular** [lattice] **law** is the distributive law restricted to non-antichain triples x, y, z.



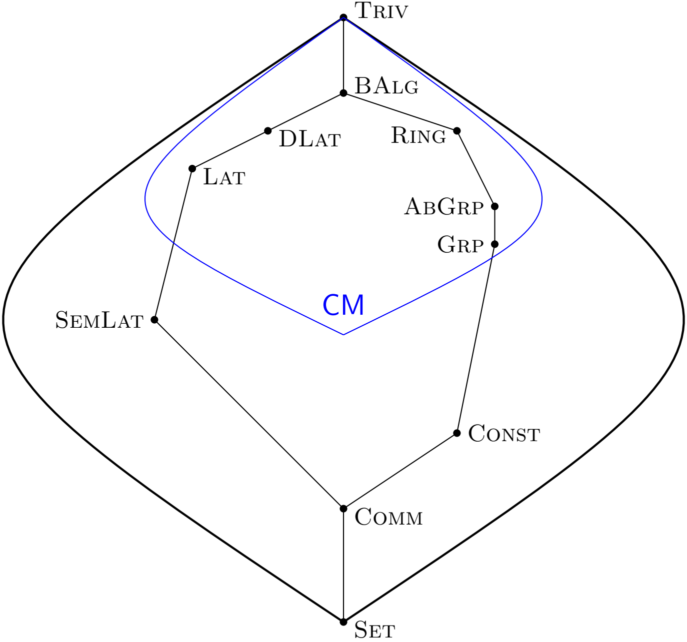
modular



not modular

Definition

A variety is **congruence modular** (CM) if all of its congruence lattices are modular.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SET

Easy Proposition

The class of congruence modular varieties forms a filter in $\mathcal{L}$.

Proof.

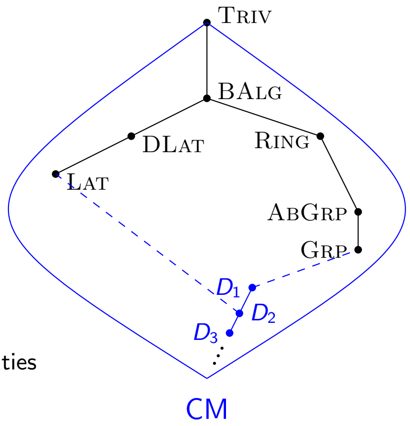
Assume $[V] \leq [W]$ and suppose V is CM.

- Fix $\mathbf{B} \in W$.
- Choose a term reduct $\mathbf{A} = (B, S)$ of $\mathbf{B}$ with $\mathbf{A} \in V$.
- $\mathrm{Con}(\mathbf{B})$ is a sublattice of $\mathrm{Con}(\mathbf{A})$.
- Modular lattices are closed under forming sublattices.
- Hence $\mathrm{Con}(\mathbf{B})$ is modular, proving W is CM.

A similar proof shows that if V, W are CM, then the canonical variety representing $[V] \wedge [W]$ is CM; the key property of modular lattices used is that they are closed under forming products. $\square$

Theorem (A. Day, 1969)

The CM filter in $\mathcal{L}$ is generated by a countable sequence $D_1, D_2, \ldots$ of finitely presented varieties; i.e., it is a Mal'cev class.



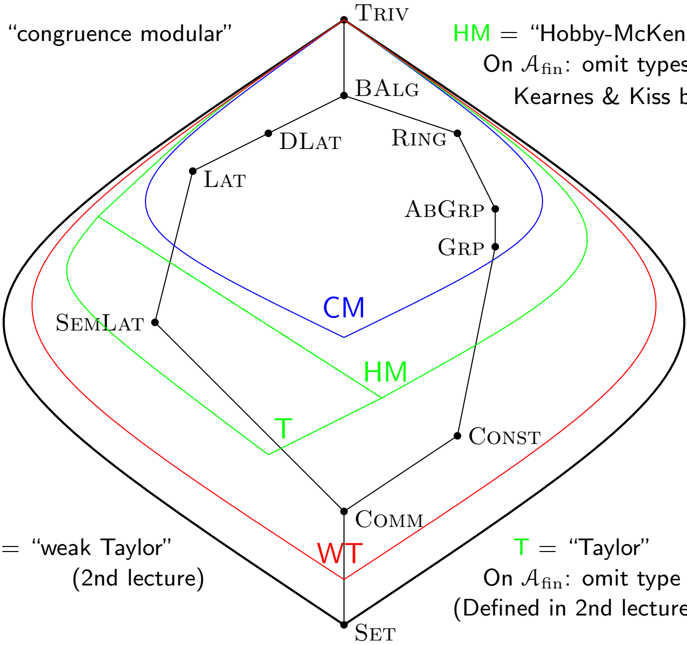
D_n has n basic operations,

defined by $2n$ simple identities

More Mal'cev classes

CM = "congruence modular"

HM = "Hobby-McKenzie"
On $\mathcal{A}_{\mathrm{fin}}$: omit types 1,5
Kearnes & Kiss book



WT = "weak Taylor"
(2nd lecture)

T = "Taylor"
On $\mathcal{A}_{\mathrm{fin}}$: omit type 1
(Defined in 2nd lecture)

Part II: finite relational structures

relational structure: a structure $\mathbf{H} = (H; \{\textit{relations}\})$.

Primitive positive (pp) formula: a first-order formula of the form $\exists \mathbf{y}[\alpha_1(\mathbf{x}, \mathbf{y}) \wedge \cdots \wedge \alpha_k(\mathbf{x}, \mathbf{y})]$ where each α_i is atomic.

- pp-formula $\varphi(\mathbf{x})$ in n free variables defines an n-ary relation $\varphi^{\mathbf{H}}$ on H.

Definition

$\mathrm{Rel}_{\mathrm{pp}}(\mathbf{H}) = \{\varphi^{\mathbf{H}} : \varphi \text{ a pp-formula in } n \geq 1 \text{ free variables}\}$.

Definition

$\mathbf{G}, \mathbf{H}$ are **pp-equivalent** if they have the same universe and the same pp-definable relations.

Definition

Given two relational structures $\mathbf{G}, \mathbf{H}$ in the languages L, L' respectively, we say that $\mathbf{G}$ is **pp-interpretable** in $\mathbf{H}$ if:
for some $k \geq 1$ there exist

1. a pp-L'-formula $\Delta(\mathbf{x})$ in k free variables;
2. a pp-L'-formula $E(\mathbf{x}, \mathbf{y})$ in $2k$ free variables;
3. for each n-ary relation symbol $R \in L$, a pp-L'-formula $\varphi_R(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ in nk free variables;

such that

4. $E^{\mathbf{H}}$ is an equivalence relation on $\Delta^{\mathbf{H}}$;
5. For each n-ary $R \in L$, $\varphi_R^{\mathbf{H}}$ is an n-ary $E^{\mathbf{H}}$-invariant relation on $\Delta^{\mathbf{H}}$;
6. $(\Delta^{\mathbf{H}}/E^{\mathbf{H}}, (\varphi_R^{\mathbf{H}}/E^{\mathbf{H}})_{R \in L})$ is isomorphic to $\mathbf{G}$.

Notation: $\mathbf{G} \prec_{\mathrm{pp}} \mathbf{H}$.

Examples

- If $\mathbf{G}$ is a reduct of $(H, \mathrm{Rel}_{\mathrm{pp}}(\mathbf{H}))$, then $\mathbf{G} \prec_{\mathrm{pp}} \mathbf{H}$.
- If $\mathbf{G}$ is a substructure of $\mathbf{H}$ and the universe of G is a pp-definable relation of $\mathbf{H}$, then $\mathbf{G} \prec \mathbf{H}$.
- For any $n \geq 3$, if $\mathbf{K}_n$ is the complete graph on n vertices, then $\mathbf{G} \prec_{\mathrm{pp}} \mathbf{K}_n$ for every **finite** relational structure $\mathbf{G}$.
- If $\mathbf{G}$ is a 1-element structure[3], then $\mathbf{G} \prec_{\mathrm{pp}} \mathbf{H}$ for every $\mathbf{H}$.

[3] Added post-lecture: and the language of **G** is empty

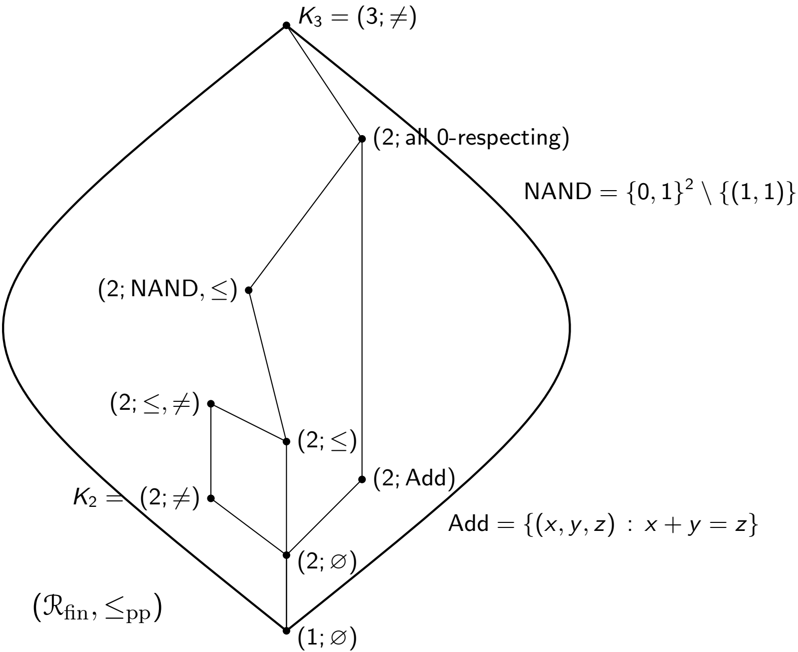
For the rest of this tutorial, we consider only **finite** relational structures (added post-lecture) all of whose fundamental relations are non-empty.

The relation $\prec_{\mathrm{pp}}$ on finite relational structures[4] is a pre-order (reflexive and transitive).

So we get a partial order in the usual way:

$$
\begin{array}{rcl}
\mathbf{G} \sim_{\mathrm{pp}} \mathbf{H} & \text{iff} & \mathbf{G} \prec_{\mathrm{pp}} \mathbf{H} \prec_{\mathrm{pp}} \mathbf{G} \\
[\mathbf{H}] & = & \{\mathbf{G} : \mathbf{G} \sim_{\mathrm{pp}} \mathbf{H}\} \\
\mathcal{R}_{\mathrm{fin}} & = & \{[\mathbf{H}] : \mathbf{H} \text{ a finite relational structure}\} \\
[\mathbf{G}] \leq_{\mathrm{pp}} [\mathbf{H}] & \text{iff} & \mathbf{G} \prec_{\mathrm{pp}} \mathbf{H}.
\end{array}
$$

[4] Added post-lecture: all of whose fundamental operations are non-empty


$K_3 = (3; \neq)$
(2; all 0-respecting)
$\mathrm{NAND} = \{0,1\}^2 \setminus \{(1,1)\}$
$(2; \mathrm{NAND}, \leq)$
$(2; \leq, \neq)$
$(2; \leq)$
$(2; \mathrm{Add})$
$K_2 = (2; \neq)$
$\mathrm{Add} = \{(x,y,z) : x + y = z\}$
$(2; \varnothing)$
$(\mathcal{R}_{\mathrm{fin}}, \leq_{\mathrm{pp}})$
$(1; \varnothing)$

Connection to algebra

Definition

Let $\mathbf{H}$ be a finite relational structure and $n \geq 1$. An ***n*-ary polymorphism of H** is a homomorphism $\mathbf{H}^n \to \mathbf{H}$.

(In particular, a unary polymorphism is an endomorphism of $\mathbf{H}$.)

Definition

Let $\mathbf{H}$ be a finite relational structure.

- $\mathrm{Pol}(\mathbf{H}) = \{$all polymorphisms of $\mathbf{H}\}$.
- The **polymorphism algebra of H** is

$$\mathrm{PolAlg}(\mathbf{H}) := (H; \mathrm{Pol}(\mathbf{H})).$$

Definition

Let $\mathbf{H}$ be a finite relational structure and V a variety of algebras. We say that **$\mathbf{H}$ admits** V if some term reduct of $\mathrm{PolAlg}(\mathbf{H})$ is in V.

Proposition (new?)

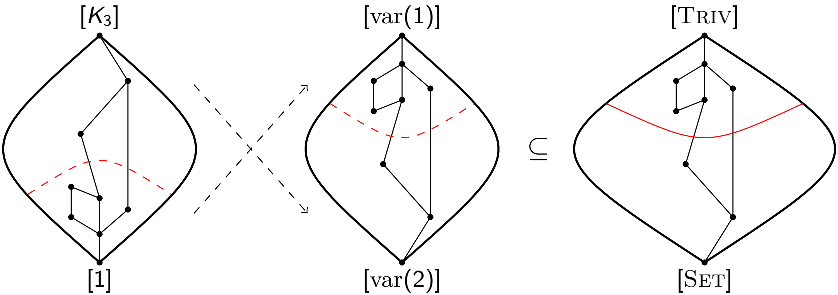
Suppose $\mathbf{G}, \mathbf{H}$ are finite relational structures. TFAE:

1. $\mathbf{G} \prec_{\mathrm{pp}} \mathbf{H}$.
2. $\mathrm{var}(\mathrm{PolAlg}(\mathbf{H})) \to \mathrm{var}(\mathrm{PolAlg}(\mathbf{G}))$.
3. $\mathbf{G}$ admits $\mathrm{var}(\mathrm{PolAlg}(\mathbf{H}))$.
4. $\mathbf{G}$ admits every finitely presented variety admitted by $\mathbf{H}$.

Corollary

The map $[\mathbf{H}] \mapsto [\mathrm{var}(\mathrm{PolAlg}(\mathbf{H}))]$ is a well-defined order anti-isomorphism from $(\mathcal{R}_{\mathrm{fin}}, \leq_{\mathrm{pp}})$ into $(\mathcal{L}, \leq)$, with image $\mathcal{A}_{\mathrm{fin}}$.

Summary



$(\mathcal{R}_{\mathrm{fin}}, \leq_{\mathrm{pp}})$ fin. rel. structures

$(\mathcal{A}_{\mathrm{fin}}, \leq)$ fin. gen'd varieties

$(\mathcal{L}, \leq)$ varieties

- Interpretation relation on varieties gives us $\mathcal{L}$.
- Sitting inside $\mathcal{L}$ is the countable $\wedge$-closed sub-poset $\mathcal{A}_{\mathrm{fin}}$.
- Pp-definability relation on finite structures gives us $\mathcal{R}_{\mathrm{fin}}$.
- $\mathcal{R}_{\mathrm{fin}}$ and $\mathcal{A}_{\mathrm{fin}}$ are anti-isomorphic
- Mal'cev classes in $\mathcal{L}$ induce filters on $\mathcal{A}_{\mathrm{fin}}$, and hence ideals on $\mathcal{R}_{\mathrm{fin}}$.