Tutorial on Universal Algebra, Mal'cev Conditions, and Finite Relational Structures: Lecture II

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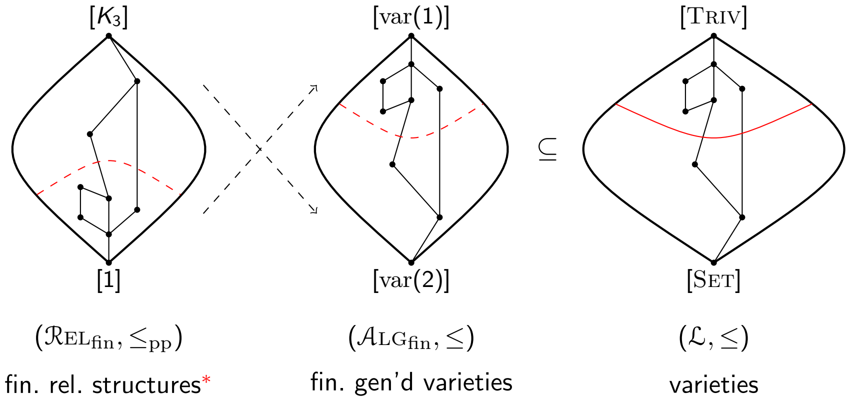
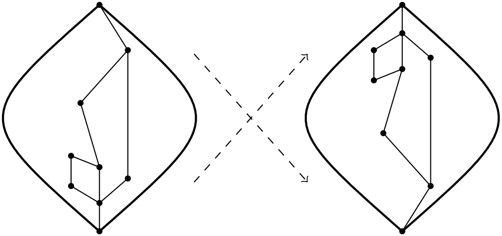
Recap

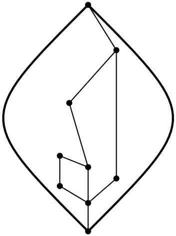


- Interpretation relation on varieties gives us $\mathcal{L}$.
- Sitting inside $\mathcal{L}$ is the countable (??) $\wedge$-closed sub-poset $\mathcal{A}\mathrm{LG}_{\mathrm{fin}}$.
- Pp-definability relation on finite structures gives us $\mathcal{R}\mathrm{EL}_{\mathrm{fin}}$.
- $\mathcal{R}\mathrm{EL}_{\mathrm{fin}}$ and $\mathcal{A}\mathrm{LG}_{\mathrm{fin}}$ are anti-isomorphic via $[\mathbf{H}] \mapsto [\mathrm{var}(\mathrm{PolAlg}(\mathbf{H}))]$.
- Mal'cev classes in $\mathcal{L}$ induce filters on $\mathcal{A}\mathrm{LG}_{\mathrm{fin}}$ and ideals on $\mathcal{R}\mathrm{EL}_{\mathrm{fin}}$.

One more set to define:

$\mathcal{R}\mathrm{EL}_{\mathrm{fin}} =$  $= \mathcal{A}\mathrm{LG}_{\mathrm{fin}}$

$\cup\!|$

$\mathcal{R}\mathrm{EL}_{\mathrm{fin}}^{\omega} :=$  $= \{[\mathbf{H}] \in \mathcal{R}\mathrm{EL}_{\mathrm{fin}}$: language of $\mathbf{H}$ is **finite**$\}$

Convention: henceforth, all mentioned relational structures under consideration have **finite** languages.

Theorem (Hell, Nešetřil, 1990)

Suppose $\mathbf{G}$ *is a finite undirected graph (without loops).*

- *If* $\mathbf{G}$ *is bipartite, then* $\mathrm{CSP}(\mathbf{G})$ *is in P.*
- *Otherwise,* $\mathrm{CSP}(\mathbf{G})$ *is NP-complete.*

What the heck is "$\mathrm{CSP}(\mathbf{G})$"?

Definition

Given a finite relational structure $\mathbf{G}$ with finite language L, the **constraint satisfaction problem with fixed template G**, written $\mathrm{CSP}(\mathbf{G})$, is the following decision problem:

Input: an arbitrary finite L-structure $\mathbf{I}$.
Question: does there exist a homomorphism $\mathbf{I} \to \mathbf{G}$?

Also called the **G-homomorphism** (or **G-coloring**) problem.

Some context

- [Classical]: $\mathrm{CSP}(K_2) \equiv$ checking bipartiteness, which is in P.
  $\mathrm{CSP}(K_n) \equiv$ graph n-colorability, which is NP-complete for $n \geq 3$ (Karp).

- **Key fact** [Essentially due to Bulatov & Jeavons, unpubl.]:

  If $\mathbf{G}, \mathbf{H}$ are finite structures in finite languages and $\mathbf{G} \prec_{\mathrm{pp}} \mathbf{H}$, then $\mathrm{CSP}(\mathbf{G})$ is no harder than $\mathrm{CSP}(\mathbf{H})$.

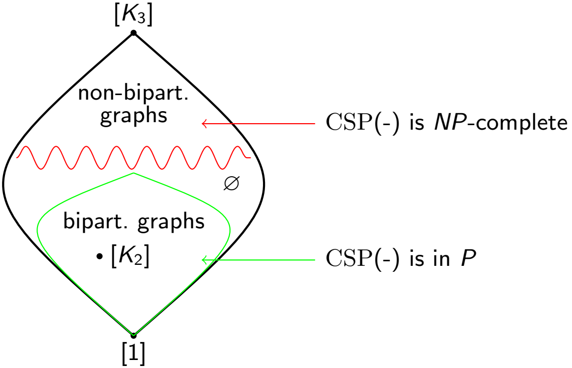
  Consequences:

  - If $\mathrm{CSP}(\mathbf{G})$ is in P [resp. NP-complete], then same is true $\forall\, \mathbf{H} \in [\mathbf{G}]$.
  - $\{[\mathbf{G}] : \mathrm{CSP}(\mathbf{G}) \text{ is in } P\}$ is a down-set in $\mathcal{R}\mathrm{EL}^{\omega}_{\mathrm{fin}}$.
  - $\{[\mathbf{G}] : \mathrm{CSP}(\mathbf{G}) \text{ is } NP\text{-complete}\}$ is an up-set in $\mathcal{R}\mathrm{EL}^{\omega}_{\mathrm{fin}}$.

- In fact:
  - $\{[\mathbf{G}] : \mathrm{CSP}(\mathbf{G}) \text{ is in } P\}$ is an ideal in $(\mathcal{R}\mathrm{EL}^{\omega}_{\mathrm{fin}}, \vee)$. (Not hard)

Pictorially:

$\mathcal{R}\mathrm{EL}^{\omega}_{\mathrm{fin}}$:



Hell-Nešetřil theorem: there is **dichotomy** for undirected graphs.

The CSP dichotomy conjecture (Feder, Vardi (1998))

There is general dichotomy. I.e., for every finite relational structure **G** in a finite language, $\mathrm{CSP}(\mathbf{G})$ is either in *P* or is *NP*-complete.

Initial steps towards a proof of the Dichotomy Conjecture

1. Reduction to cores.

Definition

Let $\mathbf{G}, \mathbf{H}$ be finite relational structures in the same language.

- $\mathbf{G}$ is **core** if all of its endomorphisms are automorphisms.
- $\mathbf{G}$ is a **core of H** if $\mathbf{G}$ is core and is a retract of $\mathbf{H}$.

Facts:

- Every finite relational structure $\mathbf{H}$ has a core, which is unique up to isomorphism; call it $\mathrm{core}(\mathbf{H})$.
- $\mathrm{CSP}(\mathbf{H}) = \mathrm{CSP}(\mathrm{core}(\mathbf{H}))$.

Hence when testing dichotomy, we need only consider cores.

2. Reduction to the endo-rigid case.

Definition

Let $\mathbf{H} = (H, \{\mathit{relations}\})$ be a relational structure.

- $\mathbf{H}$ is **endo-rigid** if its only endomorphism is id_H.
- $\mathbf{H}^c := (H, \{\mathit{relations}\} \cup \{\, \{a\} \ : \ a \in H\})$. ("**H** with constants")

Facts:

- Endo-rigid $\Rightarrow$ core.
- $\mathbf{H}^c$ is endo-rigid.

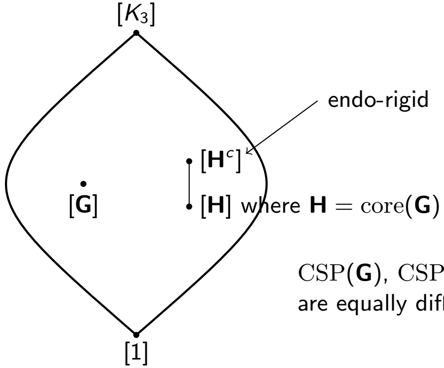
Proposition (Bulatov, Jeavons, Krokhin, 2005)

If $\mathbf{H}$ is core, then $\mathrm{CSP}(\mathbf{H})$ and $\mathrm{CSP}(\mathbf{H}^c)$ have the same difficulty.

Hence when testing general dichotomy, we need only consider structures with constants (equivalently, endo-rigid structures).

The reductions in pictures:

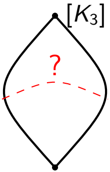
$\mathcal{R}\mathrm{EL}^{\omega}_{\mathrm{fin}}$:



$\mathrm{CSP}(\mathbf{G})$, $\mathrm{CSP}(\mathbf{H})$, and $\mathrm{CSP}(\mathbf{H}^c)$ are equally difficult.

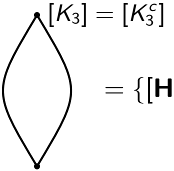
“When testing general dichotomy, we need only consider endo-rigid structures.”

$\mathcal{R}\mathrm{EL}^{\omega}_{\mathrm{fin}} =$



$\cup\!\mid$

Define $\mathcal{E} :=$

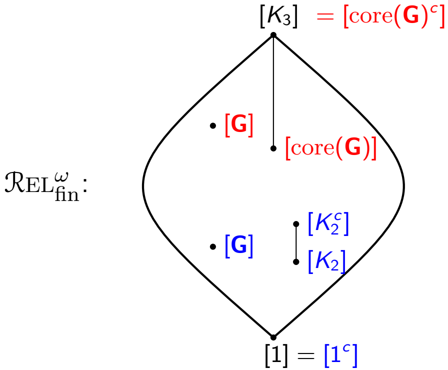


$= \{[\mathbf{H}] \in \mathcal{R}\mathrm{EL}^{\omega}_{\mathrm{fin}} \,:\, \mathbf{H} \text{ is endo-rigid}\}$

$\therefore$ To establish general dichotomy, it suffices to establish dichotomy in $\mathcal{E}$.

Question: Where in $\mathcal{E}$ should the “dividing line” be?

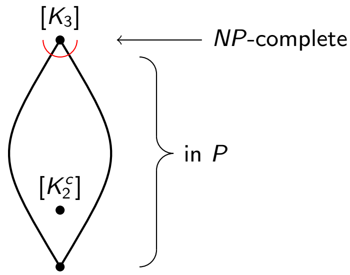
Consider the situation for graphs.



Hell-Nešetřil explained: for a finite graph $\mathbf{G}$,

- $\mathbf{G}$ bipartite $\Rightarrow \mathrm{core}(\mathbf{G}) = K_2$ or 1.
- $\mathbf{G}$ non-bipartite $\Rightarrow \ldots [\mathrm{core}(\mathbf{G})^c] = [K_3]$.

Question: Where in $\mathcal{E}$ should the "dividing line" be?

$\mathcal{E} =$ 

The Algebraic CSP Dichotomy Conjecture (BKJ 2000)

We have dichotomy in $\mathcal{E}$; moreover, the "dividing line" separating P from NP-complete is between $\mathcal{E} \setminus \{[K_3]\}$ and $\{[K_3]\}$.

Back to algebra: the **Taylor class** T.

Definition

T = the class of varieties V such that $\exists n \geq 1$, $\exists$ term $t(x_1, \ldots, x_n)$ s.t.

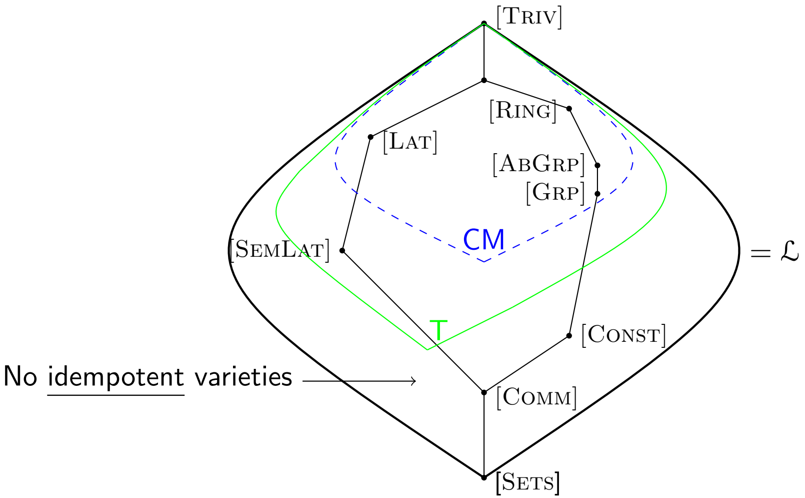
1. $\forall 1 \leq i \leq n$, $\exists$ an identity of the form

$$V \models t(\text{vars}, \underset{\substack{\uparrow \\ i}}{x}, \text{vars}) \approx t(\text{vars}, \underset{\substack{\uparrow \\ i}}{y}, \text{vars});$$

2. $V \models t(x, x, \ldots, x) \approx x$. ("$t$ is idempotent.")

Jargon: such a term t (witnessing $V \in T$) is called a **Taylor term** for V.

Fact: T forms a filter in $\mathcal{L}$ (and hence is a Mal'cev class).



Theorem (Taylor, 1977)

For any **idempotent** *variety V (i.e., all basic operations are idempotent), either $[V] = [\text{SETS}]$ or $V \in T$.*

Now suppose **H** is a finite endo-rigid structure.

Then every basic operation of $\mathrm{PolAlg}(\mathbf{H})$ is idempotent.

- PROOF: $f \in \mathsf{Pol}(\mathbf{H}) \Rightarrow f(x, x, \ldots, x)$ is an endomorphism of **H**
  $\Rightarrow f(x, x, \ldots, x) \approx x$ (**H** is endo-rigid).

Hence $V := \mathrm{var}(\mathrm{PolAlg}(\mathbf{H}))$ is an idempotent variety.

As $[\mathbf{H}] = [K_3]$ in $\mathcal{E}$ iff $[V] = [\textsc{Sets}]$ in $\mathcal{L}$, we get

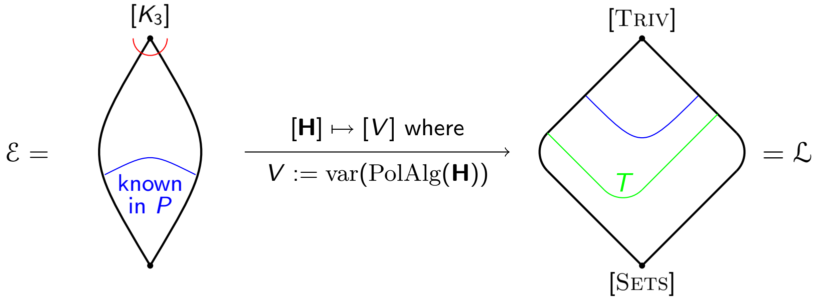
Corollary

Suppose $[\mathbf{H}] \in \mathcal{E}$.

- *If* $[\mathbf{H}] \neq [K_3]$, *then* $\mathrm{var}(\mathrm{PolAlg}(\mathbf{H})) \in T$ *(i.e.,* **H** *has a "Taylor polymorphism").*
- *Hence the Algebraic Dichotomy Conjecture is equivalent to*

  $\mathbf{H}$ *endo-rigid and has a Taylor polymorphism* $\Rightarrow \mathrm{CSP}(\mathbf{H}) \in P$.

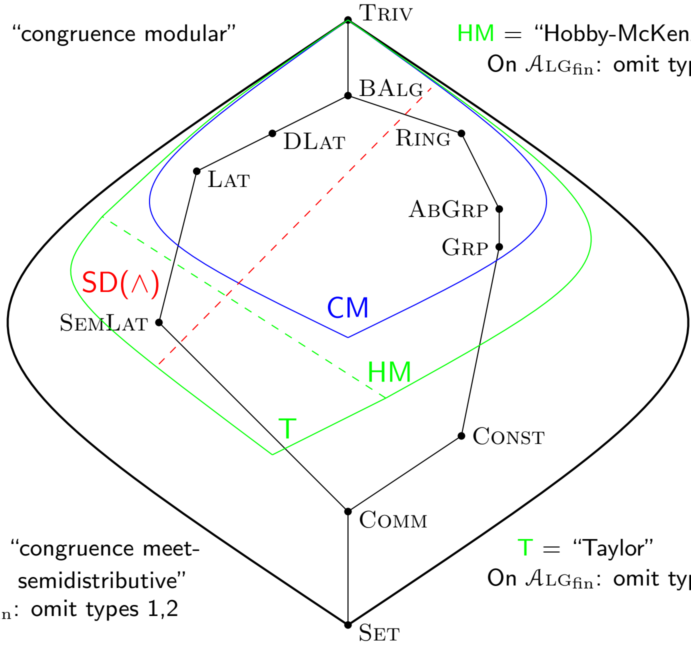
How close are we to verifying the Algebraic CSP Dichotomy Conjecture?



- Measure progress (i.e., the portion of $\mathcal{E} \setminus \{[K_3]\}$ known to be in P) via its image in $\mathcal{L}$.
- **Thesis**: progress is "robust" if its image in $\mathcal{L}$ "is" a Mal'cev class.

CM = “congruence modular”

HM = “Hobby-McKenzie”
On $\mathcal{A}\textsc{lg}_{\text{fin}}$: omit types 1,5



SD(∧) = “congruence meet-semidistributive”
On $\mathcal{A}\textsc{lg}_{\text{fin}}$: omit types 1,2

T = “Taylor”
On $\mathcal{A}\textsc{lg}_{\text{fin}}$: omit type 1

Another theme: finding "good" Taylor terms.

Definition

An operation f of arity $k \geq 2$ is called a **WNU** operation if it satisfies

$$f(y,x,x,\ldots,x) \;\approx\; f(x,y,x,\ldots,x) \;\approx\; f(x,x,y,\ldots,x) \;\approx \cdots$$

and

$$f(x,x,\ldots,x) \;\approx\; x.$$

Observe: any WNU is a Taylor operation.

Theorem (Maróti, McKenzie, 2008, verifying a conjecture of Valeriote)

Suppose $\mathbf{A}$ *is a finite algebra and* $V = \mathrm{var}(\mathbf{A})$. *If* V *has a Taylor term, then* V *has a WNU term.*

Definition

An operation f of arity $k \geq 2$ is called a **cyclic** operation if it satisfies

$$f(x_1, x_2, x_3, \ldots, x_k) \approx f(x_2, x_3, \ldots, x_k, x_1)$$

and

$$f(x, x, \ldots, x) \approx x.$$

Observe: any cyclic operation is a WNU, since we can specialize the first identity to get

$$f(y, x, x, \ldots, x) \approx f(x, y, x, \ldots, x) \approx f(x, x, y, \ldots, x) \approx \cdots.$$

Theorem (Barto, Kozik, 201?)

Suppose $\mathbf{A}$ *is a finite algebra and* $V = \mathrm{var}(\mathbf{A})$*. If* V *has a Taylor term, then* V *has a cyclic term. (In fact, has a* p*-ary cyclic term for every prime* $p > |A|$*.)*

Easy proof of the Hell-Nešetřil theorem, using cyclic terms.
(Due to Barto, Kozik?)

Let $\mathbf{G} = (G, E)$ be a finite graph; assume that it is core and not bipartite.

We must show that $[\mathbf{G}^c] = [K_3]$.

Assume the contrary. Then $\mathbf{G}^c$ (and hence also $\mathbf{G}$) has a Taylor polymorphism.

So by the Barto-Kozik theorem, $\mathbf{G}$ has a cyclic polymorphism of arity p for every prime $p > |G|$.

$\mathbf{G}$ not bipartite $\Rightarrow$ $\mathbf{G}$ contains an odd cycle, and hence contains cycles of every odd length $> |G|$.

Pick a prime $p > |G|$ and a cycle $a_1, a_2, \ldots, a_p$ in $\mathbf{G}$ of length p. That is,

$$(a_1, a_2),\ (a_2, a_3),\ \ldots, (a_{p-1}, a_p),\ (a_p, a_1) \in E.$$

Pick a cyclic polymorphism f of $\mathbf{G}$ of arity p.

Observe that if

$$\begin{aligned} \mathbf{u} &= (a_1, a_2, \ldots, a_{p-1}, a_p) \\ \mathbf{v} &= (a_2, a_3, \ldots,\ a_p,\ a_1), \end{aligned}$$

then $(\mathbf{u}, \mathbf{v})$ is an edge of $\mathbf{G}^p$.

As f is a homomorphism $\mathbf{G}^p \to \mathbf{G}$, we get that $(f(\mathbf{u}), f(\mathbf{v}))$ is an edge of $\mathbf{G}$.

But $f(\mathbf{u}) = f(\mathbf{v})$ because f is cyclic. So $(f(\mathbf{u}), f(\mathbf{v}))$ is a loop.

Contradiction!!

In conclusion:

- Good progress is being made on the CSP Dichotomy Conjecture, with essential help from universal algebra.
- The conjecture is motivating new purely algebraic conjectures, some of which have been recently proved.

Thank you!