Identifying groups of finite Morley rank with a split BN-pair of rank 1

Josh Wiscons

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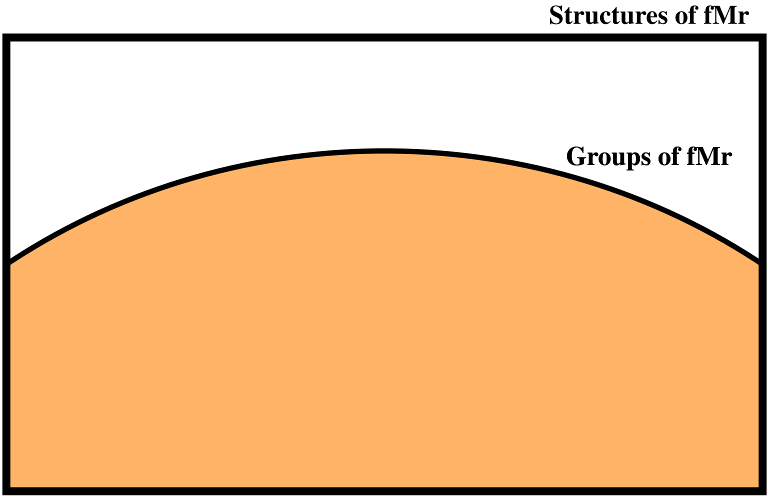
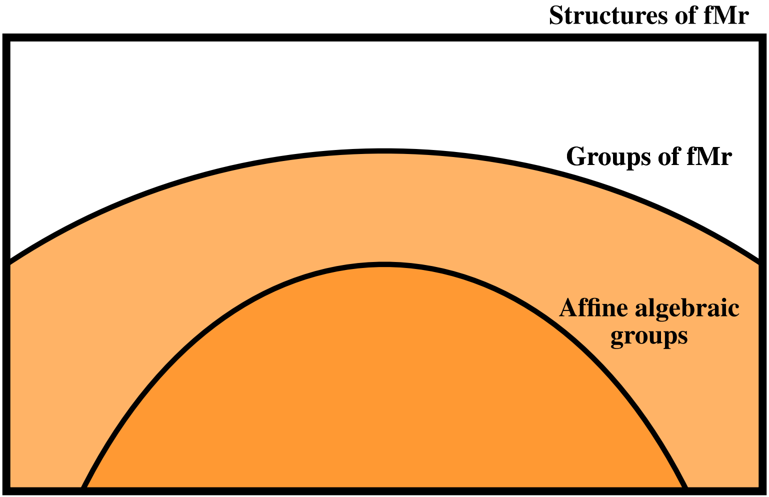
Intro: groups of finite Morley rank (fMr)




Structures of fMr
Groups of fMr
Affine algebraic groups

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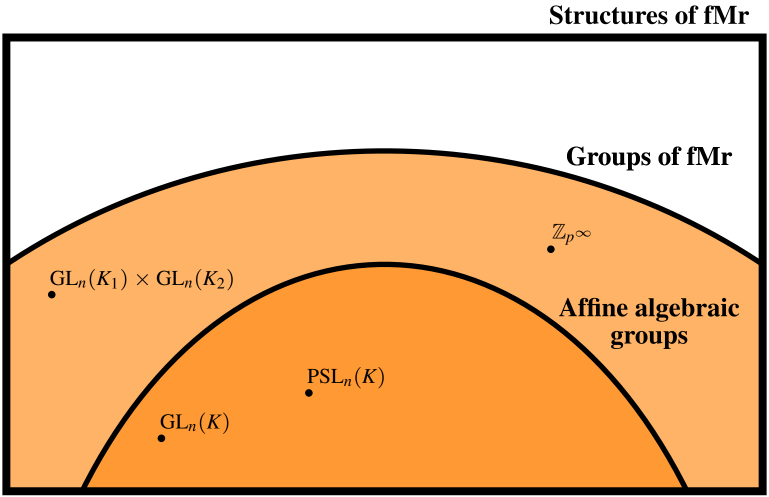

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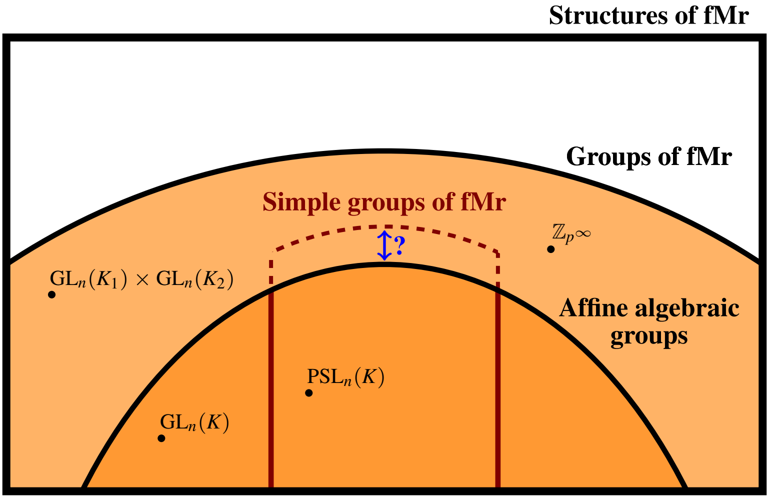

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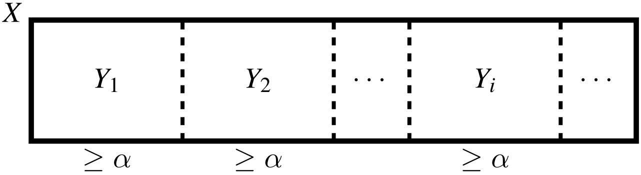

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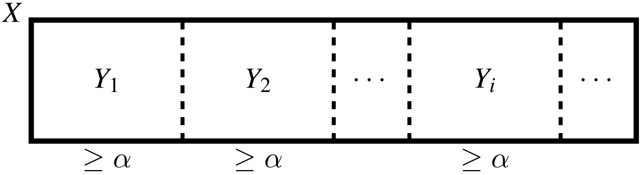

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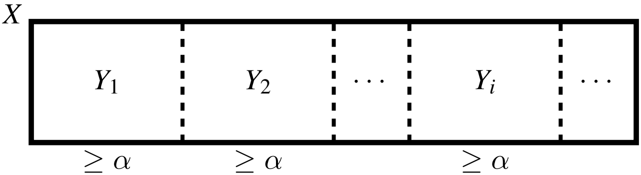

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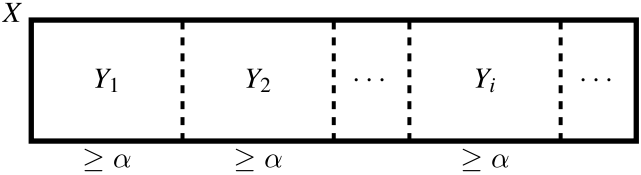

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$\mathrm{RM}(\mathcal{M})$ is defined to be $\mathrm{RM}(M)$.

Groups of finite Morley rank

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A group of finite Morley rank is an expansion of a "pure group", i.e. $(G; \cdot, ^{-1}, 1, \ldots)$, that has finite Morley rank (fMr).

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Examples

1. Affine algebraic groups over an algebraically closed field, K, have fMr: $\mathrm{GL}_n(K)$, $\mathrm{PSL}_n(K)$, finite groups, …

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2. $\mathbb{Z}$ does NOT have fMr because $\mathbb{Z} \gneqq 2\mathbb{Z} \gneqq 4\mathbb{Z} \gneqq 8\mathbb{Z} \gneqq \cdots$

Cherlin-Zil'ber Conjecture

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The analysis breaks into 4 types based on the structure of a Sylow 2-subgroup, S. The types are

- Degenerate: $S^\circ = 1$,
- Odd: S° is nontrivial, divisible, and abelian (S° is a 2-torus),
- Even: S° is nontrivial, nilpotent, and of bounded exponent (S° is 2-unipotent), and
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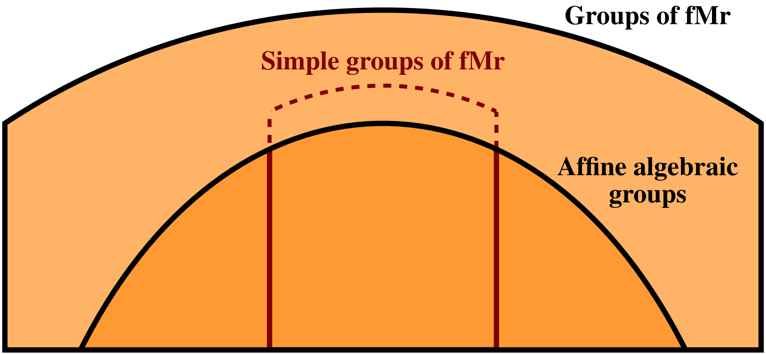
Theorem (Altınel-Borovik-Cherlin)

There are no infinite simple groups of finite Morley rank of mixed type and those of even type are indeed algebraic.

Theorem of Altınel, Borovik, and Cherlin



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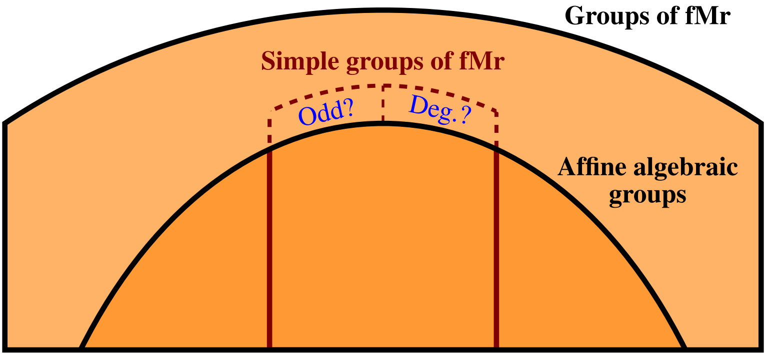

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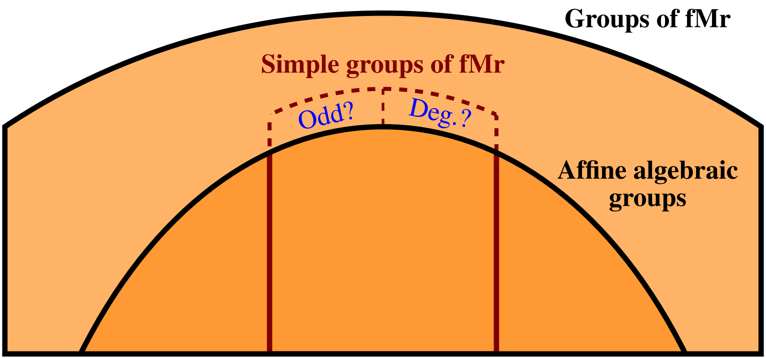
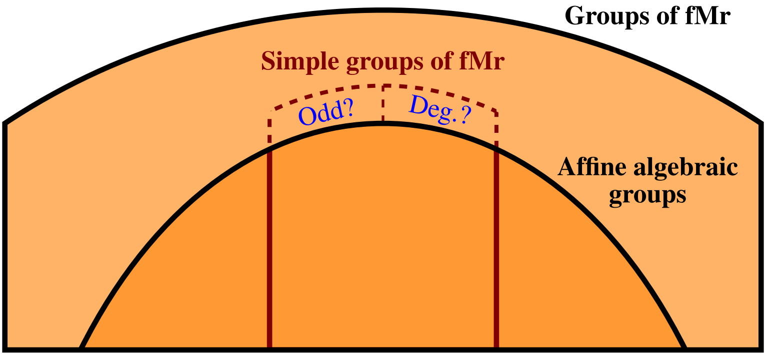


- We consider the following inductive frameworks.

  K^*-groups: groups in which every proper definable simple section is an algebraic group.



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  L^*-groups: groups in which every proper definable simple section of odd type is an algebraic group.

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Theorem (Kramer-Tent-Van Maldeghem)

An infinite simple group of finite Morley rank with an irreducible BN-pair of Tits rank at least 3 is isomorphic to an algebraic group over an algebraically closed field. The same holds for Tits rank 2 if the associated polygon is Moufang.

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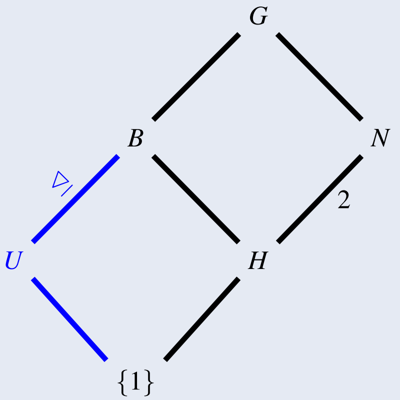
- BN-pairs for a group give rise to a geometry on which the group acts.
- In Tits rank 1, the geometry degenerates, and we simply have a group acting 2-transitively on a set.
- We focus on groups with a split BN-pair of Tits rank 1.

Split BN-pairs of Tits rank 1

General picture:



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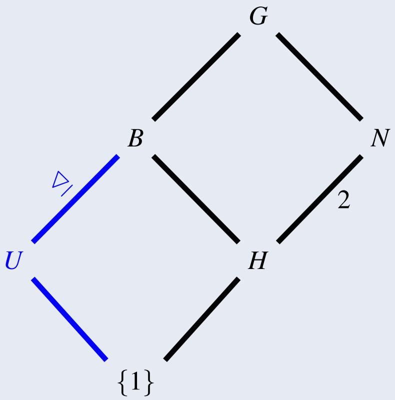

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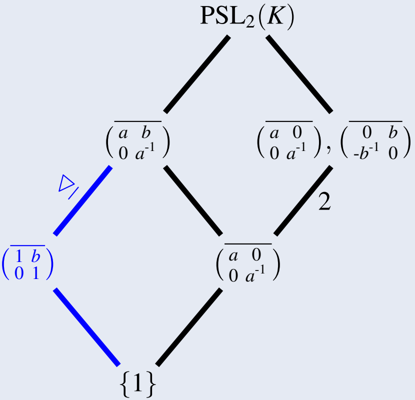

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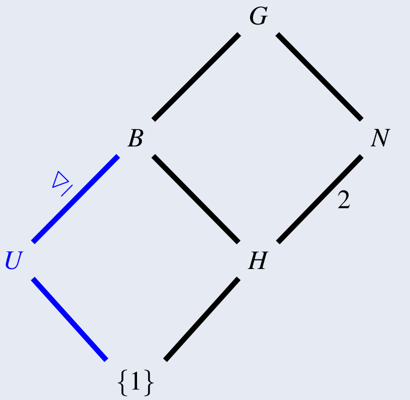

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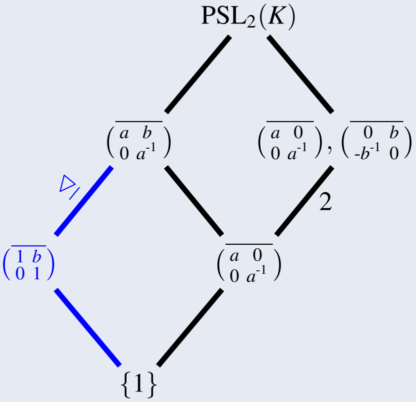

- $\mathrm{PSL}_2(K) = B \sqcup B\overline{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}B$
- $\overline{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}} \cong K^+$ and $\overline{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}} \cong K^\times$

The main result

Theorem (W)

Let G be an infinite simple L^*-group of finite Morley rank with a split BN-pair of Tits rank 1. If U is infinite and abelian, then $G \cong \mathrm{PSL}_2(K)$ for K an algebraically closed field.

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   iv. $H = Q$, so H is algebraic;
   v. a Borel subgroup of H has an infinite centralizer CONTRADICTING (i).

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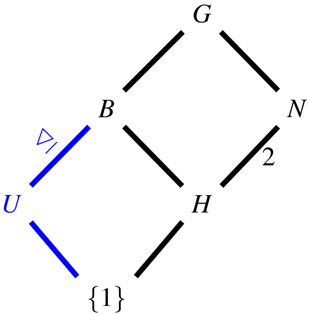

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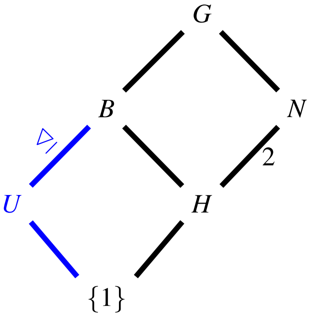

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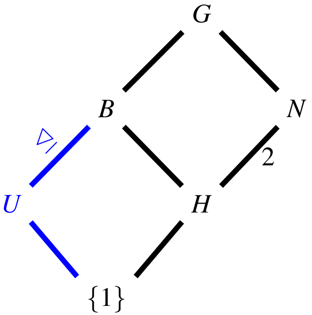

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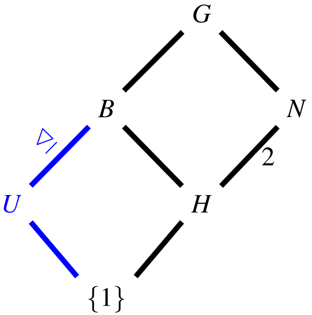

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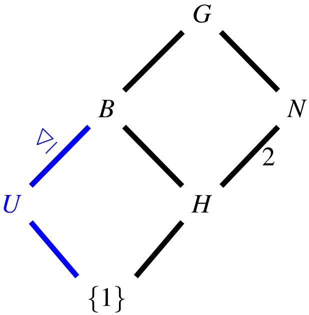

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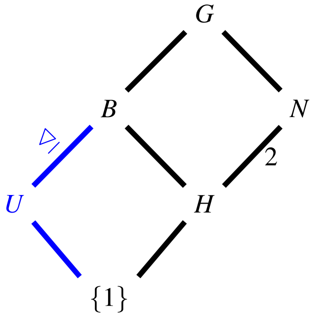

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Q2: Can we address when U is arbitrary and H is abelian?

Q3: Can we apply the main result to simple groups with Prüfer 2-rank 1?