The Poincaré Model

Axiom (B4) and Congruence for Segments and Angles

April 22, 2020
Recall: The Poincaré Model (Points, Lines, and Betweenness)

Let $\Pi_F$ be a Cartesian plane $\Pi_F$ over a Euclidean ordered field $(F; <)$, and let $\Gamma$ be a fixed circle in $\Pi_F$ with center $O$.

Definition. The points of the Poincaré model, called $P$-points, are the points of $\Pi_F$ inside $\Gamma$.

The lines of the Poincaré model, called $P$-lines, are the sets $\ell_P$ of $P$-points of lines $\ell$ (in $\Pi_F$) that pass through $O$; and the sets $\gamma_P$ of $P$-points of circles $\gamma$ (in $\Pi_F$) that are perpendicular to $\Gamma$.

Definition. For $P$-points $A$, $B$, $C$, $B$ is $P$-between $A$ and $C$, denoted $A \ast_P B \ast_P C$, if $A$, $B$, $C$ are distinct $P$-points on a $P$-line $\lambda$, and

- if $\lambda = \ell_P$ for a line $\ell$ through $O$ (in $\Pi_F$), then $A \ast_B C$ (in $\Pi_F$), while
- if $\lambda = \gamma_P$ for a circle $\gamma \perp \Gamma$ with center $\hat{O}$, (in $\Pi_F$), then $\overrightarrow{\hat{O}B}$ is in the interior of $\angle A\hat{O}C$ (in $\Pi_F$).

Theorem. In the Poincaré model, the incidence and betweenness axioms (I1)–(I3), (B1)–(B3) hold, but Playfair's axiom (P) fails.
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- the incidence and betweenness axioms (I1)–(I3), (B1)–(B3) hold,
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- the incidence and betweenness axioms (I1)–(I3), (B1)–(B3) hold,
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Recall that for a geometry satisfying (I1)–(I3), (B1)–(B3) we have that (B4) ⇔ PSThm. Hence, to show that (B4) holds in the P-model, it suffices to show that PSThm holds.

**Definition.** Define the P-sides of a P-line as follows:

- P-sides of ℓ_P: the sets of P-points on the two sides of ℓ (in Π_F);
- P-sides of γ_P: the sets of P-points inside, resp., outside γ (in Π_F).

**Plane Separation Theorem for the P-model.** For every P-line λ and for any two P-points A, B not on λ, the following are equivalent:

- (i) A, B are on different P-sides of λ;
- (ii) there is a P-point L ∈ λ such that A ∗ P L ∗ P B.

**Idea of Proof of PSThm** when λ = γ_P and the unique P-line containing A, B is δ_P (the other cases are similar, but easier). Let the centers of γ, δ be C, D. (γ_P ̸= δ_P ⇒ C ̸= D.)

- (i) ⇒ γ, δ meet at L ̸= L′ (in Π_F) [axiom (E)], L′ = ρΓ(L) [as γ, δ ⊥ Γ].
- (1) γ_P, δ_P meet at a P-point L, and (2) line CD is outside Γ (in Π_F) (so γ_P ∪ δ_P is on the same side of CD in Π_F).
- (ii) ⇒ (1), (2) as well.

Hence:

- (i) ⇔ CA < CL < CB (in Π_F) (I.24,25) ⇔ ∠CDA < ∠CDL < ∠CDB (in Π_F) ±, < for ∠s ⇔ (ii).
Recall that for a geometry satisfying (I1)–(I3), (B1)–(B3) we have that (B4) ⇔ PSThm. Hence, to show that (B4) holds in the P-model, it suffices to show that PSThm holds.

**Definition.** Define the *P-sides* of a P-line as follows:

- **P-sides of** $\ell_{P}$: the sets of P-points on the two sides of $\ell$ (in $\Pi_F$);
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- (i) $\Rightarrow$ $\gamma, \delta$ meet at $L \neq L'$ (in $\Pi_F$) \[axiom (E)\], $L' = \rho_{\Gamma}(L)$ [as $\gamma, \delta \perp \Gamma$].

- $\Rightarrow (1)$ $\gamma_P, \delta_P$ meet at a P-point $L$, and (2) line $CD$ is outside $\Gamma$ (in $\Pi_F$) (so $\gamma_P \cup \delta_P$ is on the same side of $CD$ in $\Pi_F$).

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- (i) $\iff CA <_P CL <_P CB$ (in $\Pi_F$) \[I.24, 25\] $\iff \angle CDA <_P \angle CDL <_P \angle CDB$ (in $\Pi_F$) $\pm, <$ for $\angle$s $\iff$ (ii).
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**Idea of Proof of PSThm** when $\lambda = \gamma_P$ and the unique P-line containing $A, B$ is $\delta_P$ (the other cases are similar, but easier). Let the centers of $\gamma, \delta$ be $C, D$.

- (i) $\implies \gamma, \delta$ meet at $L \neq L'$ (in $\Pi_F$) [axiom (E)], $L' = \rho_{\Gamma}(L)$ [as $\gamma, \delta \perp \Gamma$].

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- (i) $\iff CA < CL < CB$ (in $\Pi_F$) (I.24,25) $\iff \angle CDA < \angle CDL < \angle CDB$ (in $\Pi_F$) $\pm, <$ for $\angle$s $\iff$ (ii).
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**Idea of Proof of PSThm** when \( \lambda = \gamma_P \) and the unique P-line containing \( A, B \) is \( \delta_P \) (the other cases are similar, but easier). Let the centers of \( \gamma, \delta \) be \( C, D \).

- \( (i) \implies (2) \): \( \gamma, \delta \) meet at \( L \neq L' \) (in \( \Pi_F \)) \[axiom (E)\], \( L' = \rho_{\Gamma}(L) \) \[as \( \gamma, \delta \perp \Gamma \)\].
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Hence:

- \( (i) \iff CA < \lambda < CL < CB \) (in \( \Pi_F \)) \[I.24,25\] 
- \( \angle CDA < \lambda < \angle CDL < \angle CDB \) (in \( \Pi_F \)) \[\pm, < \text{for } \angle s \iff (ii)\].
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• (i) $\Rightarrow \gamma, \delta$ meet at $L \neq L'$ (in $\Pi_F$) [axiom (E)],

• (ii) $\Rightarrow (1)$ as well. Hence:

• (i) $\iff CA < CL < CB$ (in $\Pi_F$) (I.24,25) $\iff \angle CDA < \angle CDL < \angle CDB$ (in $\Pi_F$) $\pm$, $<$ for $\angle$s $\iff$ (ii).
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\[ \begin{array}{c}
\text{Diagram showing the Poincaré Disk model with P-line and points.}
\end{array} \]
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- (i) \( \Rightarrow \) \( \gamma, \delta \) meet at \( L \neq L' \) (in \( \Pi_F \)) [axiom (E)],
- \( L' = \rho_\Gamma (L) \) [as \( \gamma, \delta \perp \Gamma \)].
- \( \Rightarrow \) (1) \( \gamma_P, \delta_P \) meet at a P-point \( L \), and
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2. there is a P-point \( L \in \lambda \) such that \( A \ast_P L \ast_P B \).

**Idea of Proof of PSThm** when \( \lambda = \gamma_P \) and the unique P-line containing \( A, B \) is \( \delta_P \) (the other cases are similar, but easier). Let the centers of \( \gamma, \delta \) be \( C, D \). (\( \gamma_P \neq \delta_P \Rightarrow C \neq D \).)
- (i) \( \Rightarrow \) \( \gamma, \delta \) meet at \( L \neq L' \) (in \( \Pi_F \)) [axiom (E)],
  \[ L' = \rho_\Gamma (L) \] [as \( \gamma, \delta \perp \Gamma \)].
- \( \Rightarrow \) (1) \( \gamma_P, \delta_P \) meet at a P-point \( L \), and
  (2) line \( CD \) is outside \( \Gamma \) (in \( \Pi_F \))
  (so \( \gamma_P \cup \delta_P \) is on the same side of \( CD \) in \( \Pi_F \)).
Recall that for a geometry satisfying (I1)–(I3), (B1)–(B3) we have that \((\text{B4}) \iff \text{PSThm}\). Hence, to show that (B4) holds in the P-model, it suffices to show that PSThm holds.

**Definition.** Define the *P-sides* of a P-line as follows:

- P-sides of \(\ell_P\): the sets of P-points on the two sides of \(\ell\) (in \(\Pi_F\));
- P-sides of \(\gamma_P\): the sets of P-points inside, resp., outside \(\gamma\) (in \(\Pi_F\)).

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(i) \(A, B\) are on different P-sides of \(\lambda\);

(ii) there is a P-point \(L \in \lambda\) such that \(A \ast_P L \ast_P B\).

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- (i) \(\Rightarrow\) \(\gamma, \delta\) meet at \(L \neq L'\) (in \(\Pi_F\)) [axiom (E)],
  \[L' = \rho_{\Gamma}(L)\] [as \(\gamma, \delta \perp \Gamma\)].
  \(\Rightarrow\) (1) \(\gamma_P, \delta_P\) meet at a P-point \(L\), and
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- (ii) \(\Rightarrow\) (1), (2) as well.
Recall that for a geometry satisfying (I1)–(I3), (B1)–(B3) we have that (B4) ⇔ PSThm. Hence, to show that (B4) holds in the P-model, it suffices to show that PSThm holds.

**Definition.** Define the *P-sides* of a P-line as follows:
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**Plane Separation Theorem for the P-model.** For every P-line \( \lambda \) and for any two P-points \( A, B \) not on \( \lambda \), the following are equivalent:

(i) \( A, B \) are on different P-sides of \( \lambda \);
(ii) there is a P-point \( L \in \lambda \) such that \( A \perp_P L \perp_P B \).

**Idea of Proof of PSThm** when \( \lambda = \gamma_P \) and the unique P-line containing \( A, B \) is \( \delta_P \) (the other cases are similar, but easier). Let the centers of \( \gamma, \delta \) be \( C, D \). \((\gamma \neq \delta \Rightarrow C \neq D.)\)

- (i) \( \Rightarrow \gamma, \delta \text{ meet at } L \neq L' \) (in \( \Pi_F \)) [axiom (E)],
  \[ L' = \rho_{\Gamma}(L) \text{ [as } \gamma, \delta \perp \Gamma]. \]
  \( \Rightarrow \) (1) \( \gamma_P, \delta_P \text{ meet at a P-point } L \), and
  (2) line \( CD \) is outside \( \Gamma \) (in \( \Pi_F \))
  (so \( \gamma_P \cup \delta_P \) is on the same side of \( CD \) in \( \Pi_F \)).

- (ii) \( \Rightarrow \) (1), (2) as well.

Hence:

- (i) \( \Leftrightarrow \overline{CA} \leq \overline{CL} \leq \overline{CB} \) (in \( \Pi_F \))
Recall that for a geometry satisfying (I1)–(I3), (B1)–(B3) we have that (B4) ⇔ PSThm. Hence, to show that (B4) holds in the P-model, it suffices to show that PSThm holds.

**Definition.** Define the *P-sides* of a P-line as follows:

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**Plane Separation Theorem for the P-model.** For every P-line \( \lambda \) and for any two P-points \( A, B \) not on \( \lambda \), the following are equivalent:

(i) \( A, B \) are on different P-sides of \( \lambda \);

(ii) there is a P-point \( L \in \lambda \) such that \( A \not\in_P L \not\in_P B \).

**Idea of Proof of PSThm** when \( \lambda = \gamma_P \) and the unique P-line containing \( A, B \) is \( \delta_P \) (the other cases are similar, but easier). Let the centers of \( \gamma, \delta \) be \( C, D \). (\( \gamma_P \neq \delta_P \Rightarrow C \neq D \).)

- (i) \( \Rightarrow \gamma, \delta \) meet at \( L \neq L' \) (in \( \Pi_F \)) [axiom (E)],
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  \( \Rightarrow (1) \gamma_P, \delta_P \) meet at a P-point \( L \), and
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- (ii) \( \Rightarrow (1), (2) \) as well.

Hence:

- (i) \( \iff \triangle CA \lesssim \triangle CL \lesssim \triangle CB \) (in \( \Pi_F \))
  (I.24,25)
- \( \iff \angle CDA \lesssim \angle CDL \lesssim \angle CDB \) (in \( \Pi_F \))
Recall that for a geometry satisfying (I1)–(I3),(B1)–(B3) we have that (B4) $\Leftrightarrow$ PSThm. Hence, to show that (B4) holds in the P-model, it suffices to show that PSThm holds.

**Definition.** Define the *P-sides* of a P-line as follows:

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**Plane Separation Theorem for the P-model.** For every P-line $\lambda$ and for any two P-points $A, B$ not on $\lambda$, the following are equivalent:

(i) $A, B$ are on different P-sides of $\lambda$;

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  $L' = \rho_\Gamma (L)$ [as $\gamma, \delta \perp \Gamma$].

  $\Rightarrow$ (1) $\gamma_P, \delta_P$ meet at a P-point $L$, and
  (2) line $CD$ is outside $\Gamma$ (in $\Pi_F$)
  (so $\gamma_P \cup \delta_P$ is on the same side of $CD$ in $\Pi_F$).

- (ii) $\Rightarrow$ (1), (2) as well.

Hence:

- (i) $\Leftrightarrow$ $CA \lesssim CL \lesssim CB$ (in $\Pi_F$)

  (I.24,25)

  $\Leftrightarrow$ $\angle CDA \lesssim \angle CDL \lesssim \angle CDB$ (in $\Pi_F$) $\pm, \leq$ for $\angle s$ (ii).
Definition of Congruence for Line Segments

Segments, rays, angles, triangles in the P-model, called *P-segments*, *P-rays*, *P-angles*, *P-triangles*, are defined using P-betweenness.

\[ \mu(AB)^P = (AB, PQ)^{-1} = (AB, PQ) \]

where \( P, Q \) are obtained as follows: for the unique P-line \( \lambda = \ell_P \) or \( \gamma_P \) through \( A, B \), \( \ell, \gamma \), resp. meets \( \Gamma \) (in \( \Pi_F \)) at \( P, Q \), and the labelling is chosen so that \( P \) is closer to \( A \) than to \( B \).

Why don’t we define \( \mu(AB)^P \), without inversion, by \( \mu(AB)^P = (AB, PQ) \)?

**Example.** Let \( A, B \) be distinct P-points on a P-line \( \lambda = \ell_P \), and let \( P, Q \) be defined as before. Then, in \( \Pi_F \), \( A, B \in PQ \{ P, Q \} \), and \( A \sim B \sim Q \).

By the solution to the problem on WSH26, if \( A \) is fixed on \( \ell_P \) (i.e., \( P, Q, A \) are fixed), then in \( \Pi_F \\
\bullet A \sim B \sim Q \) if and only if \( 0 < (AB, PQ) < 1 \), and
\[ \text{as } 0 < k := (AB, PQ) < 1 \text{ increases, the length of } PB = PA + k \cdot AQ \cdot PQ \text{ decreases} \]

(and hence so does the length of \( AB \)).
Definition of Congruence for Line Segments

Segments, rays, angles, triangles in the P-model, called \( P\text{-segments} \), \( P\text{-rays} \), \( P\text{-angles} \), \( P\text{-triangles} \), are defined using P-betweenness.

**Notation:** \( \overrightarrow{AB}^P \), \( 
\overrightarrow{AB}^P \), \( \angle P\,BAC \), \( \triangle P\,ABC \).

**Definition.** For distinct P-points \( A \), \( B \), let
\[
\mu(\overrightarrow{AB}) = (\overrightarrow{AB}, \overrightarrow{PQ})^{-1} = (\overrightarrow{AB}, \overrightarrow{PQ})
\]
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**Example.** Let \( A \), \( B \) be distinct P-points on a P-line \( \lambda = \ell^P \), and let \( P \), \( Q \) be defined as before. Then, in \( \Pi^F \), \( A \), \( B \in \overrightarrow{PQ} \{P, Q\} \), and \( A^* B^* Q \).

By the solution to the problem on WSH26, if \( A \) is fixed on \( \ell^P \) (i.e., \( P \), \( Q \), \( A \) are fixed), then in \( \Pi^F \):

- \( A^* B^* Q \) if and only if \( 0 < (\overrightarrow{AB}, \overrightarrow{PQ}) < 1 \), and
- as \( 0 < k := (\overrightarrow{AB}, \overrightarrow{PQ}) < 1 \) increases, the length of \( PB = PA + k \cdot AQ \cdot PQ \) decreases (and hence so does the length of \( AB \)).

**Definition.** Two P-segments \( \overrightarrow{AB}^P \) and \( \overrightarrow{A'B'}^P \) are P-congruent, denoted \( \overrightarrow{AB}^P \sim P \overrightarrow{A'B'}^P \), if \( \mu(\overrightarrow{AB}) = \mu(\overrightarrow{A'B'}) \).
Segments, rays, angles, triangles in the P-model, called $P$-segments, $P$-rays, $P$-angles, $P$-triangles, are defined using P-betweenness.

Notation: $\overrightarrow{AB}^P$, $\overrightarrow{AB}^P$, $\angle_P BAC$, $\triangle_P ABC$.

**Definition.** For distinct P-points $A, B$, let 
\[ \mu(AB) = (AB, PQ)^{-1} = \frac{1}{(AB, PQ)} \]
where $P, Q$ are obtained as follows:

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Why don't we define $\mu(AB)$, without inversion, by $\mu(AB) = (AB, PQ)$ as above? 

**Example.** Let $A, B$ be distinct P-points on a P-line $\lambda = \ell_P$, and let $P, Q$ be defined as before. Then, in $\Pi_F$, $A, B \in PQ \setminus \{P, Q\}$, and $A \neq B \neq Q$.

By the solution to the problem on WSH26, if $A$ is fixed on $\ell_P$ (i.e., $P, Q, A$ are fixed), then in $\Pi_F$:

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Definition of Congruence for Line Segments

Segments, rays, angles, triangles in the P-model, called \textit{P-segments}, \textit{P-rays}, \textit{P-angles}, \textit{P-triangles}, are defined using P-betweenness.

Notation: $\overline{AB}^P$, $\overrightarrow{AB}^P$, $\angle_P BAC$, $\triangle_P ABC$.

**Definition.** For distinct P-points $A, B$, let $\mu(AB) = (AB, PQ)^{-1} = \frac{1}{(AB, PQ)}$ where $P, Q$ are obtained as follows: for the unique P-line $\lambda (= \ell_P$ or $\gamma_P$) through $A, B$, $\ell$, resp. $\gamma$, meets $\Gamma$ (in $\Pi_F$) at $P, Q$,

Why don't we define $\mu(AB)$, without inversion, by $\mu(AB) = (AB, PQ)$ ($P, Q$ as above)?

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The Poincaré Model
Definition of Congruence for Line Segments

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\[ \text{Why don't we define } \mu(AB), \text{ without inversion, by } \mu(AB) = (AB, PQ) \text{?} \]

\textbf{Example.} Let $A, B$ be distinct P-points on a P-line $\lambda = \ell_P$, and let $P, Q$ be defined as before. Then, in $\Pi_F$, $A, B \in PQ \{P, Q\}$, and $A \ast B \ast Q$.

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Definition of Congruence for Line Segments

Segments, rays, angles, triangles in the P-model, called $P$-segments, $P$-rays, $P$-angles, $P$-triangles, are defined using $P$-betweenness.

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**Example.** Let \( A, B \) be distinct P-points on a P-line \( \lambda = \ell_P \), and let \( P, Q \) be defined as before. Then, in \( \Pi_F \), \( A, B \in \overline{PQ} \setminus \{P, Q\} \), and \( A \ast B \ast Q \).
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Why don’t we define \( \mu(AB) \), without inversion, by \( \mu(AB) = (AB, PQ) \ (P, Q \text{ as above}) \)?

Example. Let \( A, B \) be distinct P-points on a P-line \( \lambda = \ell_P \), and let \( P, Q \) be defined as before. Then, in \( \Pi_F \), \( A, B \in \overline{PQ} \setminus \{P, Q\} \), and \( A \ast B \ast Q \). By the solution to the problem on WSH26, if \( A \) is fixed on \( \ell_P \) (i.e., \( P, Q, A \) are fixed), then in \( \Pi_F \):

- \( A \ast B \ast Q \) if and only if \( 0 < (AB, PQ) < 1 \), and
- as \( 0 < k := (AB, PQ) < 1 \) increases, the length of \( \overline{PB} = \frac{PA}{PA+k\cdot AQ} \cdot \overline{PQ} \) decreases (and hence so does the length of \( \overline{AB} \)).
Segments, rays, angles, triangles in the P-model, called \textit{P-segments}, \textit{P-rays}, \textit{P-angles}, \textit{P-triangles}, are defined using P-betweenness.

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Why don’t we define \(\mu(AB)\), without inversion, by \(\mu(AB) = (AB, PQ)\) (\(P, Q\) as above)?

\textbf{Example.} Let \(A, B\) be distinct P-points on a P-line \(\lambda = \ell_P\), and let \(P, Q\) be defined as before. Then, in \(\Pi_F\), \(A, B \in PQ \setminus \{P, Q\}\), and \(A \ast B \ast Q\). By the solution to the problem on WSH26, if \(A\) is fixed on \(\ell_P\) (i.e., \(P, Q, A\) are fixed), then in \(\Pi_F\):
- \(A \ast B \ast Q\) if and only if \(0 < (AB, PQ) < 1\), and
- as \(0 < k := (AB, PQ) < 1\) increases, the length of \(PB = \frac{PA}{PA + k \cdot AQ} \cdot PQ\) decreases (and hence so does the length of \(AB\)).

\textbf{Definition.} Two P-segments \(\overrightarrow{AB}^P\) and \(\overrightarrow{A'B'}^P\) are \textit{P-congruent}, denoted \(\overrightarrow{AB}^P \cong_P \overrightarrow{A'B'}^P\), if \(\mu(AB) = \mu(A'B')\).
Lemma.
(1) $\mu(AB) = \mu(BA)$ for any distinct P-points $A, B$.

(C2) and (C3) Hold in the P-Model
Lemma.
(1) \( \mu(AB) = \mu(BA) \) for any distinct P-points \( A, B \).
(2) \( \mu(AB) > 1 \) for any distinct P-points \( A, B \).
(C2) and (C3) Hold in the P-Model

Lemma.
(1) $\mu(AB) = \mu(BA)$ for any distinct P-points $A, B$.
(2) $\mu(AB) > 1$ for any distinct P-points $A, B$.
(3) $\mu$ behaves like a ‘multiplicative distance function’, i.e., if $A, B, C$ are P-points such that $A \triangleright_P B \triangleright_P C$, then 
   \[ \mu(AB) \cdot \mu(BC) = \mu(AC). \]
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(1) $\mu(AB) = \mu(BA)$ for any distinct P-points $A, B$.
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(3) $\mu$ behaves like a ‘multiplicative distance function’, i.e.,
if $A, B, C$ are P-points such that $A \ast_p B \ast_p C$, then
$$\mu(AB) \cdot \mu(BC) = \mu(AC).$$

Proof. (1) $\mu(AB) = (AB, PQ)^{-1} = \frac{AQ}{AP} \cdot \frac{BP}{BQ} = (BA, QP)^{-1} = \mu(BA)$. 

(C2) and (C3) Hold in the P-Model

The Poincaré Model

MATH 3210: Euclidean and Non-Euclidean Geometry
(C2) and (C3) Hold in the P-Model

Lemma.
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(2) \( \mu(AB) > 1 \) for any distinct P-points \( A, B \).
(3) \( \mu \) behaves like a ‘multiplicative distance function’, i.e., if \( A, B, C \) are P-points such that \( A \neq_P B \neq_P C \), then
\[
\mu(AB) \cdot \mu(BC) = \mu(AC).
\]

Proof. (1) \( \mu(AB) = (AB, PQ)^{-1} = \frac{AQ}{AP} \cdot \frac{BP}{BQ} = (BA, QP)^{-1} = \mu(BA) \).
(2) \( \mu(AB) = \frac{AQ}{AP} \cdot \frac{BP}{BQ} > 1 \) because \( AP < BP, BQ < AQ \).
Lemma.

(1) \( \mu(AB) = \mu(BA) \) for any distinct P-points \( A, B \).
(2) \( \mu(AB) > 1 \) for any distinct P-points \( A, B \).
(3) \( \mu \) behaves like a ‘multiplicative distance function’, i.e., if \( A, B, C \) are P-points such that \( A \sim P B \sim P C \), then
\[
\mu(AB) \cdot \mu(BC) = \mu(AC).
\]

Proof.

(1) \( \mu(AB) = (AB, PQ)^{-1} = \frac{AQ}{AP} \cdot \frac{BP}{BQ} = (BA, QP)^{-1} = \mu(BA) \).
(2) \( \mu(AB) = \frac{AQ}{AP} \cdot \frac{BP}{BQ} > 1 \) because \( AP < BP, BQ < AQ \).
(3) \( \mu(AB) \cdot \mu(BC) = (AB, PQ)^{-1} \cdot (BC, PQ)^{-1} \)
\[
= \left( \frac{AQ}{AP} \cdot \frac{BP}{BQ} \right) \cdot \left( \frac{BQ}{BP} \cdot \frac{CP}{CQ} \right) = \frac{AQ}{AP} \cdot \frac{CP}{CQ} = (AC, PQ)^{-1} = \mu(AC).\]
Lemma.
(1) \( \mu(AB) = \mu(BA) \) for any distinct P-points \( A, B \).
(2) \( \mu(AB) > 1 \) for any distinct P-points \( A, B \).
(3) \( \mu \) behaves like a ‘multiplicative distance function’, i.e., if \( A, B, C \) are P-points such that \( A \ast_P B \ast_P C \), then

\[
\mu(AB) \cdot \mu(BC) = \mu(AC).
\]

Proof. (1) \( \mu(AB) = (AB, PQ)^{-1} = \frac{AQ}{AP} \cdot \frac{BP}{BQ} = (BA, QP)^{-1} = \mu(BA) \).
(2) \( \mu(AB) = \frac{AQ}{AP} \cdot \frac{BP}{BQ} > 1 \) because \( AP < BP, BQ < AQ \).
(3) \( \mu(AB) \cdot \mu(BC) = (AB, PQ)^{-1} \cdot (BC, PQ)^{-1} \)

\[
= (\frac{AQ}{AP} \cdot \frac{BP}{BQ}) \cdot (\frac{BQ}{BP} \cdot \frac{CP}{CQ}) = \frac{AQ}{AP} \cdot \frac{CP}{CQ} = (AC, PQ)^{-1} = \mu(AC).
\]

The Definition of \( \cong_P \) (in terms of \( \mu \)) and the Lemma above immediately imply that
Lemma.

1. \( \mu(AB) = \mu(BA) \) for any distinct P-points \( A, B \).
2. \( \mu(AB) > 1 \) for any distinct P-points \( A, B \).
3. \( \mu \) behaves like a ‘multiplicative distance function’, i.e., if \( A, B, C \) are P-points such that \( A \neq P \) and \( B \neq P \), then
   \[ \mu(AB) \cdot \mu(BC) = \mu(AC). \]

Proof. (1) \( \mu(AB) = (AB, PQ)^{-1} = \frac{AQ}{AP} \cdot \frac{BP}{BQ} = (BA, QP)^{-1} = \mu(BA). \)

(2) \( \mu(AB) = \frac{AQ}{AP} \cdot \frac{BP}{BQ} > 1 \) because \( AP < BP, BQ < AQ \).

(3) \( \mu(AB) \cdot \mu(BC) = (AB, PQ)^{-1} \cdot (BC, PQ)^{-1} \)
   \[ = \left( \frac{AQ}{AP} \cdot \frac{BP}{BQ} \right) \cdot \left( \frac{BQ}{BP} \cdot \frac{CP}{CQ} \right) = \frac{AQ}{AP} \cdot \frac{CP}{CQ} = (AC, PQ)^{-1} = \mu(AC). \]

The Definition of \( \cong_P \) (in terms of \( \mu \)) and the Lemma above immediately imply that

(C2) holds: If \( AB \cong_P CD \) and \( AB \cong_P EF \), then \( CD \cong_P EF \).

Also, every P-segment is P-congruent to itself.
(C2) and (C3) Hold in the P-Model

Lemma.
(1) $\mu(AB) = \mu(BA)$ for any distinct P-points $A, B$.
(2) $\mu(AB) > 1$ for any distinct P-points $A, B$.
(3) $\mu$ behaves like a ‘multiplicative distance function’, i.e., if $A, B, C$ are P-points such that $A \ast_P B \ast_P C$, then

$$\mu(AB) \cdot \mu(BC) = \mu(AC).$$

Proof. (1) $\mu(AB) = (AB, PQ)^{-1} = \frac{AQ}{AP} \cdot \frac{BP}{BQ} = (BA, QP)^{-1} = \mu(BA)$.
(2) $\mu(AB) = \frac{AQ}{AP} \cdot \frac{BP}{BQ} > 1$ because $AP < BP, BQ < AQ$.
(3) $\mu(AB) \cdot \mu(BC) = (AB, PQ)^{-1} \cdot (BC, PQ)^{-1}$

$$= \left( \frac{AQ}{AP} \cdot \frac{BP}{BQ} \right) \cdot \left( \frac{BQ}{BP} \cdot \frac{CP}{CQ} \right) = \frac{AQ}{AP} \cdot \frac{CP}{CQ} = (AC, PQ)^{-1} = \mu(AC).$$

The Definition of $\cong_P$ (in terms of $\mu$) and the Lemma above immediately imply that

(C2) holds: If $\overline{AB}^P \cong_P \overline{CD}^P$ and $\overline{AB}^P \cong_P \overline{EF}^P$, then $\overline{CD}^P \cong_P \overline{EF}^P$.

Also, every P-segment is P-congruent to itself.

(C3) holds: If $A, B, C$ and $A', B', C'$ are P-points s.t. $A \ast_P B \ast_P C, A' \ast_P B' \ast_P C'$, $\overline{AB}^P \cong_P \overline{A'B'}^P, \overline{BC}^P \cong_P \overline{B'C'}^P$, then $\overline{AC}^P \cong_P \overline{A'C'}^P$. 

The Poincaré Model
Definition. Two P-angles are *P-congruent* if the angles formed by their tangent rays are congruent in $\Pi_F$.

It follows easily from the Definition that (C5) holds: For any three P-angles $\alpha_P, \beta_P, \beta'_P$, if $\alpha_P \sim_P = \beta_P$ and $\alpha_P \sim_P = \beta'_P$, then $\beta_P \sim_P = \beta'_P$. Also, any P-angle is P-congruent to itself.

(C4) holds: For any P-angle $\alpha_P$, for any P-ray $\overrightarrow{DF_P}$, and for each P-side of the P-line $\lambda = DF_P$ there exists a unique P-ray $\overrightarrow{DE_P}$ on the given P-side of $\lambda$ such that $\alpha_P \sim_P = \angle_P EDF_P$.

Idea of Proof. In $\Pi_F$:
- $\exists$ unique ray on the given side which forms angle $\alpha$ with the tangent ray to $\overrightarrow{DF_P}$.
- $\exists$ unique circle $\gamma$ tangent to this ray, and passing through $D$ and $D' = \rho(\gamma, D)$.

This yields the unique P-ray $\overrightarrow{DE_P}$ on $\gamma_P$ satisfying the requirements.
**Definition.** Two $P$-angles are $P$-congruent if the angles formed by their tangent rays are congruent in $\Pi_F$. 

It follows easily from the definition that (C5) holds: For any three $P$-angles $\alpha_P, \beta_P, \beta'_P$, if $\alpha_P \sim P \beta_P$ and $\alpha_P \sim P \beta'_P$, then $\beta_P \sim P \beta'_P$. Also, any $P$-angle is $P$-congruent to itself.

(C4) holds: For any $P$-angle $\alpha_P$, for any $P$-ray $\vec{DF}_P$, and for each $P$-side of the $P$-line $\lambda = DF_P$ there exists a unique $P$-ray $\vec{DE}_P$ on the given $P$-side of $\lambda$ such that $\alpha_P \sim P \angle PEDF$. 

**Idea of Proof.** In $\Pi_F$: 

- $\exists$ unique ray on the given side which forms angle $\alpha$ with the tangent ray to $\vec{DF}_P$.
- $\exists$ unique circle $\gamma$ tangent to this ray, and passing through $D$ and $D' = \rho(\gamma(D))$.

This yields the unique $P$-ray $\vec{DE}_P$ on $\gamma_P$ satisfying the requirements.
Definition. Two P-angles are \textit{P-congruent} if the angles formed by their tangent rays are congruent in $\Pi_F$. (The tangent ray of a P-ray $\overrightarrow{AB}$ lying on a P-line $\ell_P$ is $\overrightarrow{AB}$.)
**Definition.** Two P-angles are *P-congruent* if the angles formed by their tangent rays are congruent in $\Pi_F$. (The tangent ray of a P-ray $\overrightarrow{AB}^P$ lying on a P-line $\ell_P$ is $\overrightarrow{AB}$.)

It follows easily from the Definition that

(C4) holds: For any P-angle $\alpha_P$, for any P-ray $\overrightarrow{DF}^P$, and for each P-side of the P-line $\lambda = \overrightarrow{DF}^P$ there exists a unique P-ray $\overrightarrow{DE}^P$ on the given P-side of $\lambda$ such that $\alpha_P \sim_P \angle PEDF$. 

Idea of Proof. In $\Pi_F$:

- $\exists$ unique ray on the given side which forms angle $\alpha$ with the tangent ray to $\overrightarrow{DF}^P$.
- $\exists$ unique circle $\gamma$ tangent to this ray, and passing through $D$ and $D' = \rho \Gamma(D)$.

This yields the unique P-ray $\overrightarrow{DE}^P$ on $\gamma$ satisfying the requirements.
Definition. Two P-angles are *P-congruent* if the angles formed by their tangent rays are congruent in $\Pi_F$. (The tangent ray of a P-ray $\overrightarrow{AB}$ lying on a P-line $\ell_P$ is $\overrightarrow{AB}$.)

It follows easily from the Definition that

**(C5) holds:** For any three P-angles $\alpha_P, \beta_P, \beta'_P$, if $\alpha_P \cong_P \beta_P$ and $\alpha_P \cong_P \beta'_P$, then $\beta_P \cong_P \beta'_P$. Also, any P-angle is P-congruent to itself.
**Definition.** Two P-angles are *P-congruent* if the angles formed by their tangent rays are congruent in $\Pi_F$. (The tangent ray of a P-ray $\overrightarrow{AB}^P$ lying on a P-line $\ell_P$ is $\overrightarrow{AB}$.)

It follows easily from the Definition that

**(C5) holds:** For any three P-angles $\alpha_P, \beta_P, \beta'_P$, if $\alpha_P \cong_P \beta_P$ and $\alpha_P \cong_P \beta'_P$, then $\beta_P \cong_P \beta'_P$. Also, any P-angle is P-congruent to itself.

**(C4) holds:** For any P-angle $\alpha_P$, for any P-ray $\overrightarrow{DF}^P$, and for each P-side of the P-line $\lambda = DF^P$.
**Definition.** Two \( P \)-angles are \emph{\( P \)-congruent} if the angles formed by their tangent rays are congruent in \( \Pi_F \). (The tangent ray of a \( P \)-ray \( \overrightarrow{AB} \) lying on a \( P \)-line \( \ell_P \) is \( \overrightarrow{AB} \).)

It follows easily from the Definition that

\textbf{(C5) holds:} For any three \( P \)-angles \( \alpha_P, \beta_P, \beta_P' \), if \( \alpha_P \cong_P \beta_P \) and \( \alpha_P \cong_P \beta_P' \), then \( \beta_P \cong_P \beta_P' \). Also, any \( P \)-angle is \( P \)-congruent to itself.

\textbf{(C4) holds:} For any \( P \)-angle \( \alpha_P \), for any \( P \)-ray \( \overrightarrow{DF} \), and for each \( P \)-side of the \( P \)-line \( \lambda = DF_P \) there exists a unique \( P \)-ray \( \overrightarrow{DE} \) on the given \( P \)-side of \( \lambda \) such that \( \alpha_P \cong_P \angle_P EDF \).
Definition of Congruence for Angles; (C4) and (C5) Hold

**Definition.** Two P-angles are *P-congruent* if the angles formed by their tangent rays are congruent in $\Pi_F$. (The tangent ray of a P-ray $\overrightarrow{AB}$ lying on a P-line $l_P$ is $\overrightarrow{AB}$.)

It follows easily from the Definition that

(C5) holds: For any three P-angles $\alpha_P, \beta_P, \beta'_P$, if $\alpha_P \cong_P \beta_P$ and $\alpha_P \cong_P \beta'_P$, then $\beta_P \cong_P \beta'_P$. Also, any P-angle is P-congruent to itself.

(C4) holds: For any P-angle $\alpha_P$, for any P-ray $\overrightarrow{DF}$, and for each P-side of the P-line $\lambda = DF^P$ there exists a unique P-ray $\overrightarrow{DE}$ on the given P-side of $\lambda$ such that $\alpha_P \cong_P \angle_P EDF$.

**Idea of Proof.** In $\Pi_F$:

- $\exists$ unique ray on the given side which forms angle $\alpha$ with the tangent ray to $\overrightarrow{DF}$.
**Definition.** Two P-angles are *P-congruent* if the angles formed by their tangent rays are congruent in $\Pi_F$. (The tangent ray of a P-ray $\overrightarrow{AB}$ lying on a P-line $\ell_P$ is $\overrightarrow{AB}$.)

It follows easily from the Definition that

(C5) holds: For any three P-angles $\alpha_P, \beta_P, \beta'_P$, if $\alpha_P \cong \beta_P$ and $\alpha_P \cong \beta'_P$, then $\beta_P \cong \beta'_P$. Also, any P-angle is P-congruent to itself.

(C4) holds: For any P-angle $\alpha_P$, for any P-ray $\overrightarrow{DF}$, and for each P-side of the P-line $\lambda = DF^P$ there exists a unique P-ray $\overrightarrow{DE}$ on the given P-side of $\lambda$ such that $\alpha_P \cong \angle_P EDF$.

**Idea of Proof.** In $\Pi_F$:
- $\exists$ unique ray on the given side which forms angle $\alpha$ with the tangent ray to $\overrightarrow{DF}$.
- $\exists$ unique circle $\gamma$ tangent to this ray, and passing through $D$ and $D' = \rho_F(D)$.
Definition. Two P-angles are *P-congruent* if the angles formed by their tangent rays are congruent in $\Pi_F$. (The tangent ray of a P-ray $\overrightarrow{AB}$ lying on a P-line $\ell_P$ is $\overrightarrow{AB}$.)

It follows easily from the Definition that

(C5) holds: For any three P-angles $\alpha_P, \beta_P, \beta'_P$, if $\alpha_P \cong_P \beta_P$ and $\alpha_P \cong_P \beta'_P$, then $\beta_P \cong_P \beta'_P$. Also, any P-angle is P-congruent to itself.

(C4) holds: For any P-angle $\alpha_P$, for any P-ray $\overrightarrow{DF}$, and for each P-side of the P-line $\lambda = DF^P$ there exists a unique P-ray $\overrightarrow{DE}$ on the given P-side of $\lambda$ such that $\alpha_P \cong_P \angle_P EDF$.

Idea of Proof. In $\Pi_F$:

- $\exists$ unique ray on the given side which forms angle $\alpha$ with the tangent ray to $\overrightarrow{DF}$.
- $\exists$ unique circle $\gamma$ tangent to this ray, and passing through $D$ and $D' = \rho_\Gamma(D)$.

This yields the unique P-ray $\overrightarrow{DE}$ on $\gamma_P$ satisfying the requirements.
Problem. Let $\Gamma$ be the circle with equation $x^2 + y^2 = 1$ in $\Pi_R$.
In the Poincaré model for $\Gamma$, consider the P-points
$A = (0, 0)$ and $B = (b, 0)$ ($0 < b < 1$).

(1) Find $\mu(AB)$.

(2) Find a P-point $M$ on $\overline{AB}$ such that $\overline{AM} \cong_P \overline{MB}$ (a P-midpoint of $\overline{AB}$).