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Set Theory Homework 1
Problem 5

We claim that $\{\mathbf{Cmpr}, \mathbf{Pset}\} \vdash \mathbf{Pset}^\sharp$.

PROOF: First let's write down the formal formulas for the above:

We let $\mathbf{C} \equiv \forall x \forall w_1 \exists y \forall z (z \in y \leftrightarrow (z \in x \wedge z \subseteq w_1))$ and note that $\mathbf{C}$ is a member of $\mathbf{Cmpr}$.

We have $\mathbf{Pset} \equiv \forall A \exists Z \forall x (x \subseteq A \rightarrow x \in Z)$ and $\mathbf{Pset}^\sharp \equiv \forall A \exists P \forall x (x \in P \leftrightarrow x \subseteq A)$.

We apply metatheorem 3.11(i) to conclude that it suffices to show $\{\mathbf{C}, \mathbf{Pset}\} \vdash \mathbf{Pset}^\sharp$.

Note that no variables occur free in the formulas $\mathbf{C}$ and $\mathbf{Pset}$. In particular the variable A does not occur free. So by the generalization theorem, if we can show that

$$\{\mathbf{C}, \mathbf{Pset}\} \vdash \exists P \forall x (x \in P \leftrightarrow x \subseteq A)$$

then we will have $\{\mathbf{C}, \mathbf{Pset}\} \vdash \mathbf{Pset}^\sharp$. Let $\varphi \equiv \exists P \forall x (x \in P \leftrightarrow x \subseteq A)$.

We will use the following metatheorem:

FACT 1: Let Γ be a set of formulas and let α, β, γ be formulas. If $\Gamma \vdash \alpha$, $\Gamma \vdash \beta$ and $\{\alpha, \beta\} \vdash \gamma$, then $\Gamma \vdash \gamma$.

PROOF: By applying the deduction theorem to $\{\alpha, \beta\} \vdash \gamma$ twice, we obtain $\emptyset \vdash \alpha \rightarrow \beta \rightarrow \gamma$. Using metatheorem 3.11(i), we therefore have $\Gamma \vdash \alpha \rightarrow \beta \rightarrow \gamma$. So let the sequence $(\gamma_1, ..., \alpha \rightarrow \beta \rightarrow \gamma)$ be a Γ deduction. By hypothesis, we also have Γ deductions: $(\alpha_1, ..., \alpha)$ and $(\beta_1, ..., \beta)$. Then

$$(\alpha_1, ..., \alpha, \beta_1, ..., \beta, \gamma_1, ..., \alpha \rightarrow \beta \rightarrow \gamma)$$

is a Γ deduction. But then

$$(\alpha_1, ..., \alpha, \beta_1, ..., \beta, \gamma_1, ..., \alpha \rightarrow \beta \rightarrow \gamma, \beta \rightarrow \gamma, \gamma)$$

is a Γ deduction where we have applied modus ponens in the last two steps. This finishes the proof. Note that this also implies that if $\Gamma \vdash \alpha$ and $\alpha \vdash \gamma$, then $\Gamma \vdash \gamma$ as this is just the case where $\alpha \equiv \beta$.

We have $\{\mathbf{C}, \mathbf{Pset}\} \vdash \mathbf{C}$ (a one line deduction).
Also we have $\{\mathbf{C}, \mathbf{Pset}\} \vdash \exists Z \forall x (x \subseteq A \rightarrow x \in Z)$ by the following deduction:

(1) $\forall A \exists Z \forall x (x \subseteq A \rightarrow x \in Z)$ by hypothesis as this formula is $\mathbf{Pset}$.
(2) $\forall A \exists Z \forall x (x \subseteq A \rightarrow x \in Z) \rightarrow \exists Z \forall x (x \subseteq A \rightarrow x \in Z)$ by Ax 2.
(3) $\exists Z \forall x (x \subseteq A \rightarrow x \in Z)$ by (1), (2) and modus ponens.

By FACT 1, it now suffices to show that $\{\mathbf{C}, \exists Z \forall x (x \subseteq A \rightarrow x \in Z)\} \vdash \varphi$. We may now introduce a new constant symbol c to our language and apply existential instantiation. So it suffices to show that $\{\mathbf{C}, \forall x (x \subseteq A \rightarrow x \in c)\} \vdash \varphi$.

Now note that $\{\mathbf{C}, \forall x(x \subseteq A \to x \in c)\} \vdash \forall x(x \subseteq A \to x \in c)$ (another one line deduction). We also have $\{\mathbf{C}, \forall x(x \subseteq A \to x \in c)\} \vdash \exists y \forall z(z \in y \leftrightarrow (z \in c \wedge z \subseteq A))$ by the following deduction:

(1) $\forall x \forall w_1 \exists y \forall z(z \in y \leftrightarrow (z \in x \wedge z \subseteq w_1))$ by hypothesis as this formula is $\mathbf{C}$.
(2) $\forall x \forall w_1 \exists y \forall z(z \in y \leftrightarrow (z \in x \wedge z \subseteq w_1)) \to \forall w_1 \exists y \forall z(z \in y \leftrightarrow (z \in c \wedge z \subseteq w_1))$ by Ax. 2.
(3) $\forall w_1 \exists y \forall z(z \in y \leftrightarrow (z \in c \wedge z \subseteq w_1))$ by (1), (2) and modus ponens.
(4) $\forall w_1 \exists y \forall z(z \in y \leftrightarrow (z \in c \wedge z \subseteq w_1)) \to \exists y \forall z(z \in y \leftrightarrow (z \in c \wedge z \subseteq A))$ by Ax. 2.
(5) $\exists y \forall z(z \in y \leftrightarrow (z \in c \wedge z \subseteq A))$ by (3), (4) and modus ponens.

Note in step (4) that we substituted A for w_1, which is valid as no quantifier $\forall A$ has a free occurence of w_1 in its scope (in fact $\forall A$ doesn't occur in our formula at all). Now by FACT 1 it suffices to show that

$$\{\exists y \forall z(z \in y \leftrightarrow (z \in c \wedge z \subseteq A)), \forall x(x \subseteq A \to x \in c)\} \vdash \varphi.$$

Now we introduce another constant $d \neq c$ to our language and apply existential instantiation again. So it suffices to show that

$$\{\forall z(z \in d \leftrightarrow (z \in c \wedge z \subseteq A)), \forall x(x \subseteq A \to x \in c)\} \vdash \varphi.$$

As a first step, we have

$$\{\forall z(z \in d \leftrightarrow (z \in c \wedge z \subseteq A)), \forall x(x \subseteq A \to x \in c)\} \vdash x \in d \leftrightarrow x \subseteq A$$

by the following deduction:

(1) $\forall z(z \in d \leftrightarrow (z \in c \wedge z \subseteq A))$ by hypothesis.
(2) $\forall z(z \in d \leftrightarrow (z \in c \wedge z \subseteq A)) \to (x \in d \leftrightarrow (x \in c \wedge x \subseteq A))$ by Ax. 2 (this is valid as $\forall x$ does not occur in the formula, so a free occurence of z does not occur in the scope of a $\forall x$ quantifier).
(3) $x \in d \leftrightarrow (x \in c \wedge x \subseteq A)$ by (1), (2) and modus ponens.
(4) $\forall x(x \subseteq A \to x \in c)$ by hypothesis.
(5) $\forall x(x \subseteq A \to x \in c) \to (x \subseteq A \to x \in c)$ by Ax. 2.
(6) $x \subseteq A \to x \in c$ by (4), (5) and modus ponens.

Letting $\alpha \equiv x \subseteq A$, $\beta \equiv x \in c$ and $\gamma \equiv x \in d$ for readability, we continue our deduction:

(7) $(\alpha \to \beta) \to [(\gamma \leftrightarrow (\beta \wedge \alpha)) \to (\gamma \leftrightarrow \alpha)]$ tautology (Ax. 1).
(8) $(\gamma \leftrightarrow (\beta \wedge \alpha)) \to (\gamma \leftrightarrow \alpha)$ by (6), (7) and modus ponens as (6) is $\alpha \to \beta$.
(9) $\gamma \leftrightarrow \alpha$ by (3), (8) and modus ponens as (3) is $\gamma \leftrightarrow (\beta \wedge \alpha)$.

This finishes the deduction as $\gamma \leftrightarrow \alpha \equiv x \in d \leftrightarrow x \subseteq A$. Now observe that x does not occur free in any formula in $\{\forall z(z \in d \leftrightarrow (z \in c \wedge z \subseteq A)), \forall x(x \subseteq A \to x \in c)\}$, so that by the generalization theorem, we have

$$\{\forall z(z \in d \leftrightarrow (z \in c \wedge z \subseteq A)), \forall x(x \subseteq A \to x \in c)\} \vdash \forall x(x \in d \leftrightarrow x \subseteq A).$$

If we can show that $\forall x(x \in d \leftrightarrow x \subseteq A) \vdash \varphi$, then we are done by applying FACT 1 again. To show this, note that by definition of $\exists$, it suffices to show that

$$\forall x(x \in d \leftrightarrow x \subseteq A) \vdash \neg\forall P \neg\forall x(x \in P \leftrightarrow x \subseteq A).$$

It suffices by the contraposition metatheorem to show that

$$\forall P \neg\forall x(x \in P \leftrightarrow x \subseteq A) \vdash \neg\forall x(x \in d \leftrightarrow x \subseteq A).$$

Here is a deduction that shows the above:

(1) $\forall P \neg\forall x(x \in P \leftrightarrow x \subseteq A)$ by hypothesis.
(2) $\forall P \neg\forall x(x \in P \leftrightarrow x \subseteq A) \rightarrow \neg\forall x(x \in d \leftrightarrow x \subseteq A)$ by Ax. 2.
(3) $\neg\forall x(x \in d \leftrightarrow x \subseteq A)$ by (1), (2) and modus ponens.

This concludes the proof that $\{\mathbf{Cmpr}, \mathbf{Pset}\} \vdash \mathbf{Pset}^\sharp$.