Set Theory (MATH 6730)

HOMEWORK 2

(First draft is due on March 17, 2021)

Problems:

1. **Raymond**
   Prove that the following statements are equivalent in $\mathsf{ZFC} \setminus \{\mathsf{Fnd}\}$ (ZFC without the Axiom of Foundation):
   (a) Fnd (the Axiom of Foundation);
   (b) there is no function f with domain ω such that $f(n+1) \in f(n)$ for all $n \in \omega$.

2. **Dale**
   Prove the following statement in ZF
   For every finite set A of nonempty sets there exists a choice function for A.

3. **Mateo**
   (i) Prove in ZF that $\omega \times \omega$ and $n \times \omega$ ($n \in \omega \setminus \{0\}$) are equipotent with ω.
   (ii) Prove in ZF that the union of a finite set of countable sets is countable. (You may use the statement proved in Problem 2.)
   (iii) Modify your arguent for (ii) to prove in ZFC that the union of a countable set of countable sets is countable.

4. **Connor**
   Prove that for every infinite set A, the set of all bijections $A \to A$ has cardinality $2^{|A|}$.

5. **Nick**
   Find all ordered pairs (κ, λ) of cardinal numbers such that their sum $\kappa +_o \lambda$ in the ordinal sense equals their sum $\kappa +_c \lambda$ in the cardinal sense.

6. **Toby**
   Let μ be an infinite cardinal, and let $\prec$ be the well-ordering of $\mu \times \mu$ we used earlier[1], which is defined as follows: for any $\delta, \varepsilon, \delta', \varepsilon' \in \mu$,

$$
\begin{aligned}
(\delta, \varepsilon) \prec (\delta', \varepsilon') \qquad \Longleftrightarrow \qquad & \max(\delta, \varepsilon) < \max(\delta', \varepsilon'), \text{ or} \\
& \max(\delta, \varepsilon) = \max(\delta', \varepsilon') \text{ and } \delta < \delta', \text{ or} \\
& \max(\delta, \varepsilon) = \max(\delta', \varepsilon'),\ \delta = \delta' \text{ and } \varepsilon < \varepsilon'.
\end{aligned}
$$

   (i) Show that $(\mu \times \mu, \prec)$ and $(\mu, <)$ are isomorphic well-ordered sets.
   (ii) Use the statement in part (i) and the General Associative Law for cardinal multipliction to prove Theorem 4.7 on the handout “The Axiom of Choice. Cardinals and Cardinal Arithmetic”.

[1]See the proof of Theorem 4.1 on the handout “The Axiom of Choice. Cardinals and Cardinal Arithmetic”.

7. Chase

Prove that GCH is equivalent (in ZFC) to the following statement:

(∗) $\kappa^{\mathrm{cf}(\kappa)} = \kappa^+$ for all infinite cardinals κ.

Please write your solution in TEX, include your name at the top of the first page, and email your solution to me (`szendrei@colorado.edu`) as a pdf file so that the file name includes the string

hw2pr$k$

where k is the problem number.