Nick Jamesson
Set Theory Homework 2
Problem 5

We will find all pairs of cardinals (κ, λ) such that $\kappa +_o \lambda = \kappa +_c \lambda$ where $+_o$ denotes ordinal addition and $+_c$ denotes cardinal addition. First, we note that if either κ or λ is 0, then ordinal and cardinal addition agree. Indeed, we know $\kappa +_o 0 = \kappa$ by theorem 5.3 and $0 +_o \lambda = \lambda$ by theorem 5.4 (iii). Also, prop 11.22 in Monk shows that if both λ and κ are finite, then ordinal and cardinal addition agree. So we may assume one of κ, λ is infinite and that $\kappa \geq 1$ and $\lambda \geq 1$ to check the remaining cases.

Suppose $\kappa \geq \lambda$. We claim that $\kappa +_o \lambda \neq \kappa +_c \lambda$. Note that if κ were finite, then λ would be finite, contradicting our assumptions. So κ is infinite, and therefore $\kappa +_c \lambda = \max(\kappa, \lambda) = \kappa$. On the other hand, $\kappa +_o \lambda \geq \kappa +_o 1 > \kappa$ since $\lambda \geq 1$. The leftmost inequality follows from Th. 9.21 (ii) in Monk. Therefore $\kappa +_o \lambda > \kappa +_c \lambda$.

For the remaining case, we suppose $\kappa < \lambda$. In this case we claim that $\kappa +_o \lambda = \kappa +_c \lambda$. If λ were finite, then κ would also be finite, a contradiction to our assumptions. So λ is infinite, which implies that $\kappa +_c \lambda = \max(\kappa, \lambda) = \lambda$. Now since λ is a cardinal, it is a limit ordinal, hence we have

$$(*) \qquad \kappa +_o \lambda = \bigcup_{\alpha < \lambda} (\kappa +_o \alpha).$$

We want to show that $\lambda = \kappa +_o \lambda$. We have $\lambda \leq \kappa +_o \lambda$ (by Th. 9.21 (v) in Monk) so it suffices to show that $\kappa +_o \lambda \leq \lambda$. Let $\beta \in \kappa +_o \lambda$ so that it suffices to show that $\beta \in \lambda$. From $(*)$, we know that $\beta \in \kappa +_o \alpha$ for some $\alpha < \lambda$. Suppose towards a contradiction that $\lambda \leq \beta$. Then $\lambda \leq \beta < \kappa +_o \alpha \Rightarrow \lambda \leq \kappa +_o \alpha$. Therefore $\lambda = |\lambda| \leq |\kappa +_o \alpha|$.

Now we claim that $|\kappa +_o \alpha| = |\kappa| +_c |\alpha|$. If we have this claim and both κ and α are finite, then $|\kappa +_o \alpha|$ is finite and $\lambda \leq |\kappa +_o \alpha|$, which gives us our contradiction. If we have this claim and one of κ, α is infinite, then $|\kappa +_o \alpha| = \max(|\kappa|, |\alpha|)$. But we have both $\kappa < \lambda$ and $\alpha < \lambda$, so because λ is a cardinal, we have $|\kappa| < \lambda$ and $|\alpha| < \lambda$. This gives us our desired contradiction again. So if we can prove the claim, we are done.

We can use transfinite induction to prove the claim. Fix κ and note that $|\kappa +_o 0| = |\kappa| = |\kappa| +_c 0$.

Now suppose that $|\kappa +_o \gamma| = |\kappa| +_c |\gamma|$. Recall that for finite ordinals ξ, we know that $|\xi +_o 1| = |\xi| +_c 1$ and that for infinite ordinals ξ, we have $|\xi +_o 1| = |\xi| = |\xi| +_c 1$. That is, $|\xi +_o 1| = |\xi| +_c 1$ for any ordinal ξ. Therefore

$$\begin{aligned} |\kappa +_o (\gamma +_o 1)| &= |(\kappa +_o \gamma) +_o 1)| \\ &= |\kappa +_o \gamma| +_c 1 \\ &= |\kappa| +_c |\gamma| +_c 1 \quad \text{by the inductive hypothesis} \\ &= |\kappa| +_c |\gamma +_o 1|. \end{aligned}$$

Now suppose δ is a limit ordinal and for all $\gamma < \delta$ that $|\kappa +_o \gamma| = |\kappa| +_c |\gamma|$. We want to show that

$|\kappa +_o \delta| = |\kappa| +_c |\delta|$. We have

$$\begin{aligned} |\kappa +_o \delta| &= \Big| \bigcup_{\gamma<\delta} \kappa +_o \gamma \Big| \\ &\leq \sum_{\gamma<\delta} |\kappa +_o \gamma| \\ &= \sum_{\gamma<\delta} (|\kappa| +_c |\gamma|) \quad \text{by the inductive hypothesis .} \end{aligned}$$

Now note that if $\kappa \geq \delta$, then $|\kappa| +_c |\delta| = |\kappa|$ and also $|\kappa| +_c |\gamma| = |\kappa|$ for all $\gamma < \delta$. So the above relations tell us that $|\kappa +_o \delta| \leq \sum_{\gamma<\delta} |\kappa| = |\kappa| \cdot |\delta| = \max(|\kappa|, |\delta|) = |\kappa|$. But $\kappa \leq \kappa +_o \delta$ by Th. 9.21 (ii) in Monk. Therefore, $|\kappa| \leq |\kappa +_o \delta|$ so that $|\kappa| = |\kappa +_o \delta| \Rightarrow |\kappa| +_c |\delta| = |\kappa +_o \delta|$ as desired.

On the other hand, if $\kappa < \delta$, then $|\kappa| \leq |\delta|$. Hence $|\kappa| +_c |\delta| = |\delta|$ and $|\kappa| +_c |\gamma| \leq |\delta|$ for $\gamma < \delta$. So the above relations imply that $|\kappa +_o \delta| \leq \sum_{\gamma<\delta} |\delta| = |\delta| \cdot |\delta| = |\delta|$. But we have $\delta \leq \kappa +_o \delta$ by Th 9.21 (v) in Monk. Therefore $|\delta| \leq |\kappa +_o \delta|$ so that $|\delta| = |\kappa +_o \delta|$. Therefore $|\kappa +_o \delta| = |\kappa| +_c |\delta|$ as desired. This concludes the proof of the claim.

We summarize the results as follows: if (κ, λ) is an ordered pair of cardinals, then $|\kappa| +_o |\lambda| = |\kappa| +_c |\lambda|$ iff one of the following hold:

(1) One of κ or λ is 0.
(2) Both κ and λ are finite.
(3) λ is infinite and $\kappa < \lambda$.