SET THEORY HOMEWORK 2

CHASE MEADORS

Problem (7). Prove that GCH is equivalent (in ZFC) to the following statement:
(∗) $\kappa^{\mathrm{cf}(\kappa)} = \kappa^+$ for all infinite cardinals κ.

In ZFC we have, for any infinite cardinal κ,

$$\kappa^+ \leq \kappa^{\mathrm{cf}(\kappa)} \leq 2^\kappa (= \kappa^\kappa)$$

It's clear that if GCH holds, then all inequalities collapse and $\kappa^+ = \kappa^{\mathrm{cf}(\kappa)} = 2^\kappa$ for all κ; thus GCH $\to$ (∗).

Conversely, suppose (∗). For any regular κ, we again have $\kappa^+ = \kappa^{\mathrm{cf}(\kappa)} = 2^\kappa$, since $\mathrm{cf}(\kappa) = \kappa$, and GCH holds for κ. It remains to show GCH holds for the singular cardinals as well. So suppose κ is singular, and let $\langle \lambda_\alpha : \alpha < \mathrm{cf}(\kappa) \rangle$ be a sequence of infinite successor cardinals so that $\kappa = \sum_{\alpha < \mathrm{cf}(\kappa)} \lambda_\alpha$. Then we have

$$2^\kappa = 2^{\sum_{\alpha<\mathrm{cf}(\kappa)} \lambda_\alpha} = \prod_{\alpha<\mathrm{cf}(\kappa)} 2^{\lambda_\alpha} =^1 \prod_{\alpha<\mathrm{cf}(\kappa)} \lambda_\alpha^{\mathrm{cf}(\lambda_\alpha)} =^2 \prod_{\alpha<\mathrm{cf}(\kappa)} \lambda_\alpha^+ \leq^3 \prod_{\alpha<\mathrm{cf}(\kappa)} \kappa = \kappa^{\mathrm{cf}(\kappa)} = \kappa^+$$

Here, (1) holds because since the λ_α are successor cardinals and thus regular. Of course (2) holds by (∗). Finally, (3) holds since κ is singular and necessarily a limit cardinal; in particular, $\lambda^+ < \kappa$ for any $\lambda < \kappa$. So in fact $2^\kappa = \kappa^+$ and GCH holds for a singular κ as well; thus (∗) $\to$ GCH.