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Set Theory Homework 3
Problem 2

Assume that $\mathcal{A}$ is a Dedekind finite set of pairwise disjoint Dedekind finite sets. Our goal is to prove in **ZF** that $\bigcup \mathcal{A}$ is Dedekind finite. We will use problem 1, which states that a set A is Dedekind infinite iff there exists an injection $\omega \to A$. We establish the following basic lemma first:

LEMMA 1: If S is a Dedekind finite set and T is a set and there is an injection $f : T \to S$, then T is Dedekind finite. In particular, subsets of Dedekind finite sets are Dedekind finite.

PROOF: Suppose towards a contradiction that T is Dedekind infinite. So there exists and injection $g : \omega \to T$. But then $f \circ g : \omega \to S$ is an injection, which is a contradiction by problem 1.

We now claim the following:

CLAIM 1: If $\bigcup \mathcal{A}$ is Dedekind infinite, then ω can be written as $\omega = \bigcup \mathcal{B}$, where $\mathcal{B}$ is a Dedekind finite set of pairwise disjoint Dedekind finite sets.

PROOF: Let $f : \omega \to \bigcup \mathcal{A}$ be an injection so that $f : \omega \to f[\omega]$ is a bijection. Now for each $a \in \mathcal{A}$, we know that $a \cap f[\omega]$ is Dedekind finite by LEMMA 1. Then the preimage $f^{-1}[a \cap f[\omega]]$ is also Dedekind finite by LEMMA 1 as f is a bijection. For $b \in \mathcal{A}$, if $b \neq a$ we have $f^{-1}[a \cap f[\omega]] \cap f^{-1}[b \cap f[\omega]] = \emptyset$: otherwise there is some $n \in \omega$ with $f(n) \in a$ and $f(n) \in b$ which is a contradiction as f is a function and a and b are disjoint.

Now define:

$$\mathcal{B} = \{s \in P(\omega) : s = f^{-1}[a \cap f[\omega]] \text{ for some } a \in \mathcal{A} \text{ with } a \cap f[\omega] \neq \emptyset\}.$$

Then $\bigcup \mathcal{B} = \omega$ by construction. Also every element of $\mathcal{B}$ is Dedekind finite, and distinct elements of $\mathcal{B}$ are disjoint as shown above. We want to show that $\mathcal{B}$ is also Dedekind finite. We first claim that $\mathcal{A}' = \{a \cap f[\omega] : a \in \mathcal{A} \text{ with } a \cap f[\omega] \neq \emptyset\}$ is Dedekind finite. To see this, let $a \cap f[\omega], b \cap f[\omega] \in \mathcal{A}'$ with $a, b \in \mathcal{A}$. If $a \neq b$, we have $a \cap f[\omega] \neq b \cap f[\omega]$ since a and b are disjoint and both $a \cap f[\omega]$ and $b \cap f[\omega]$ are nonempty. We then have a function $\mathcal{A}' \to \mathcal{A}$ given by $a \cap f[\omega] \mapsto a$. This function is easily seen to be injective: if $a = b$, then $a \cap f[\omega] = b \cap f[\omega]$. Now by LEMMA 1 we know $\mathcal{A}'$ is Dedekind finite.

To show $\mathcal{B}$ is Dedekind finite, we define a function $g : \mathcal{B} \to \mathcal{A}'$ by $g(b) = f[b]$. Suppose $b, c \in \mathcal{B}$ and $g(b) = g(c)$. We have $b = f^{-1}[a \cap f[\omega]]$ and $c = f^{-1}[a' \cap f[\omega]]$ for some $a, a' \in \mathcal{A}$. And since f is a bijection (in particular a surjection) $g(b) = f[f^{-1}[a \cap f[\omega]]] = a \cap f[\omega]$. Similarly $g(c) = a' \cap f[\omega]$. If $a \neq a'$, then a and a' are disjoint so that $(a \cap f[\omega]) \cap (a' \cap f[\omega]) = \emptyset$. But $a \cap f[\omega]$ and $a' \cap f[\omega]$ are nonempty (by definition of $\mathcal{B}$) and $a \cap f[\omega] = a' \cap f[\omega]$ so we have a contradiction. Therefore $a = a'$ so that $b = c$ which shows that g is an injection. Therefore $\mathcal{B}$ is Dedekind finite by LEMMA 1 as desired. We have now proved CLAIM 1.

Now we suppose towards a contradiction that $\bigcup \mathcal{A}$ is Dedekind infinite. To derive a contradiction and hence show that $\bigcup \mathcal{A}$ is Dedekind finite, we first prove the following:

LEMMA 2: Let $S \subseteq \omega$. Then S is Dedekind finite iff S is finite (equipotent with some $n \in \omega$).

PROOF: First suppose S is not finite. Then S is not equipotent with any $n \in \omega$. But S is well ordered, and therefore order isomorphic with an ordinal α. Hence S is equipotent with α which implies $\alpha \notin \omega$. That is, $\omega \leq \alpha$. Now there is an injection of ω into α, and hence an injection of ω into S which shows S is Dedekind infinite.

On the other hand, suppose S is finite and that S is equipotent with $n \in \omega$. If S were Dedekind infinite, we would have an injection of ω into S and therefore we would have an injection of ω into n. But $n \subseteq \omega$ so there is an injection of n into ω. Then by the Schröder–Bernstein theorem (which holds in **ZF**), we see that ω is equipotent with n. But the fact that ω and n are cardinals holds in **ZF** so we have a contradiction.

We now know from LEMMA 2 that the elements of $\mathcal{B}$ are finite. Furthermore, we may define a function $h : \mathcal{B} \to \omega$ by setting $h(b)$ equal to the least element of b (recall that b is a subset of ω). Note the function h is not defined using the axiom of choice. The fact that the elements of $\mathcal{A}$ are pairwise disjoint allowed us to prove that the elements of $\mathcal{B}$ are pairwise disjoint. Therefore, we may not have $h(b) = h(b')$ for $b \neq b'$, which shows that h is injective. Without the assumption of pairwise disjoint elements, the axiom of choice would be required to obtain an injection $\mathcal{B} \to \omega$.

Finally we observe that $h : \mathcal{B} \to h[\mathcal{B}]$ is bijective so that $h[\mathcal{B}]$ is a Dedkind finite subset of ω by LEMMA 1. Then by LEMMA 2 $h[\mathcal{B}]$ is finite so that $\mathcal{B}$ is finite. We have shown that $\omega = \bigcup \mathcal{B}$ is a finite union of finite subsets of ω. To get a contradiction, we just need to prove the following in **ZF**:

CLAIM 2: If $\mathcal{B}$ is a finite set whose elements are finite subsets of ω then $\bigcup \mathcal{B}$ is finite.

PROOF: Since $\mathcal{B}$ is equipotent with a natural number, $\mathcal{B}$ has a cardinality and we may use ordinary induction on $|\mathcal{B}|$. If $|\mathcal{B}| = 0$, then the result is immediate as $\emptyset = \bigcup \mathcal{B}$. For the inductive case, suppose that $|\mathcal{B}| = n + 1$ so that we may write $\mathcal{B} = \{b_1, ..., b_{n+1}\}$. Then $\bigcup \mathcal{B} = (\bigcup \{b_1, ..., b_n\}) \cup b_{n+1}$. But $\{b_1, ..., b_n\}$ is equipotent with n and is therefore finite and b_{n+1} is finite by assumption. By the inductive hypothesis, we know $\bigcup \{b_1, ..., b_n\}$ is finite. Since finite subsets of ω have upper bounds in ω, we know there are natural numbers N and M such that $\bigcup \{b_1, ..., b_n\} \subseteq N$ and $b_{n+1} \subseteq M$. Hence $\bigcup \mathcal{B} = (\bigcup \{b_1, ..., b_n\}) \cup b_{n+1} \subseteq M \cup N$. But $M \cup N$ is a natural number (which is finite). Now LEMMA 2 implies $M \cup N$ is Dedekind finite, so LEMMA 1 implies $\bigcup \mathcal{B}$ is Dedekind finite. Applying LEMMA 2 again tells us $\bigcup \mathcal{B}$ is finite.