Set Theory HW 3.5

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April 2021

Problem 1. *Let* $(S, \subset)$ *be the Aronszajn tree constructed in the proof of Theorem* 9 *in the the lecture notes "Trees." Show that there exists a strict order preserving function* $f : S \to \mathbb{R}$.

Let $(\mathcal{P}(\omega), \subseteq)$ be the poset of subsets of ω ordered by inclusion.

Theorem 2. *There exists a strict order preserving function* $g : S \to \mathcal{P}(\omega)$.

Proof. Recall that each element of S is an injective function from a countable ordinal into ω. We define $g : S \to \mathcal{P}(\omega)$ so that

$$g(s) = \operatorname{rng}(s).$$

Suppose $s, s' \in S$ and $s \subsetneq s'$. Then the injectivity of s' implies

$$g(s) = \operatorname{rng}(s) \subsetneq \operatorname{rng}(s') = g(s').$$

□

Theorem 3. *There exists a strict order preserving function* $h : \mathcal{P}(\omega) \to \mathbb{R}$.

Proof. Given $T \subseteq \omega$, let $\mathbf{1}_T : \omega \to \{0, 1\}$ be the characteristic function of T. We define $h : \mathcal{P}(\omega) \to \mathbb{R}$ given by

$$h(T) = \sum_{n=0}^{\infty} \mathbf{1}_T(n) 2^{-n}.$$

Then clearly $h(T) \in \mathbb{R}$ and if $T \subsetneq T'$ then $h(T) < h(T')$. □

Corollary 4. *There exists a strict order preserving function* $f : S \to \mathbb{R}$.

Proof. Let $g : S \to \mathcal{P}(\omega)$ and $h : \mathcal{P}(\omega) \to \mathbb{R}$ be as given in Theorems 2 and 3. Then $h \circ g : S \to \mathbb{R}$ is a strict order preserving function from S to $\mathbb{R}$. □