HOMEWORK 3

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Theorem 1. *Let* $(\mathbb{R}, \leq)$ *be the set of reals with the usual ordering. We will say that a subset* B *of* $\mathbb{R}$ *is well-ordered if the restriction of* $<$ *to* B *is a well-order on* B. *Let us fix an* n*-ary function* $f : \mathbb{R}^n \to \mathbb{R}$ $(n \in \omega \setminus \{0\})$ *which is (weak) order preserving; i.e., satisfies* $f(x_0, \dots, x_{n-1}) \leq f(y_0, \dots, y_{n-1})$ *whenever* $x_0, \dots, x_{n-1}, y_0, \dots, y_{n-1} \in \mathbb{R}$ *are such that* $x_0 \leq y_0, \dots, x_{n-1} \leq y_{n-1}$.

Prove that if $A_0, \dots, A_{n-1}$ *are well-ordered subsets of* $\mathbb{R}$, *then the subset*

$$f[A_0, \dots, A_{n-1}] = \{f(a_0, \dots, a_{n-1}) : a_0 \in A_0, \dots, a_{n-1} \in A_{n-1}\}$$

of $\mathbb{R}$ *is also well-ordered.*

In order to prove theorem 1, we will first prove a helping lemma.

Lemma 1.1. *Given a linear order* $\prec$ *on a set* A, A *is well ordered under* $\prec$ *if, and only if,* A *contains no* $\prec$*-strictly decreasing* ω *sequences.*

Proof. To prove the forwards direction, assume that $(A, \prec)$ is a well order and, for contradiction, that there exists a decreasing ω-sequence $(a_n)_{n\in\omega}$ contained in A. Now take the set $B = \{a \in A : \exists n \in \omega (a = a_n)\}$. Since the range of a non-empty function cannot be empty, B is non-empty. Thus, by assumption, there exists a $\prec$-minimal element m of B. By construction $m = a_n$ for some $n \in \omega$. But, we have that $a_{n+1} \in B$ as well and, by assumption $a_{n+1} \prec a_n = m$, a contradiction. Thus, A contains no $\prec$-decreasing ω-sequence.

Now, to prove the converse we will prove its contrapositive, i.e. that if A is not well ordered under $\prec$, then A contains some $\prec$-decreasing ω-sequence. So, assume A is not well ordered. Since $\prec$ is linear on A, there must exist some non-empty $B \subseteq A$ where B contains no minimal element. Since B is non-empty, we may take some $b \in B$. Before defining the sequence, we invoke the choice function principle. Let C be a choice function on $\mathscr{P}(B) - \{\varnothing\}$. Now define $G : \omega \times V \to V$ by

$$G(n, v) = \begin{cases} b & \text{if } n = 0 \\ C(\{x \in B : x \prec v(m)\}) & \text{if } n = m+1,\ v \text{ is a function w/dmn} = n \text{ and } v(m) \in B \\ \varnothing & \text{otherwise} \end{cases}$$

We know that $\{x \in B : x \prec v(m)\}$ is always non-empty under the given conditions since B contains no minimal elements. The General Recursion Theorem implies there exists $F : \omega \to V$ such that $F(n) = G(n, F\restriction_n)$, for all $n \in \omega$. Clearly F is an ω-sequence. We will show the sequence $(F(n))_{n\in\omega}$ is contained in A and strictly decreasing. Consider $S = \{n \in \omega : F(n) \in B\}$. Clearly $F(0) = b \in B$ so

$0 \in S$. Assume $n \in S$ and consider $F(n+1) = G(n+1, F\restriction_{n+1})$. By the inductive hypothesis $F(n) \in B$, so $G(n+1, F\restriction_{n+1}) = C(\{x \in B : x \prec F(n)\})$. Clearly $C(\{x \in B : x \prec F(n)\}) = F(n+1) \in B$ and so $n+1 \in S$. Thus, by induction, $S = \omega$ and the sequence $(F(n))_{n\in\omega}$ is contained in B, implying it is contained in A. To see that this sequence is strictly decreasing, we will use induction on n to show that $F(n) \prec F(m)$ for all $m < n$. Consider the set $S = \{n \in \omega : \forall m < n(F(n) \prec F(m))\}$. Vacuously, $0 \in S$. Now assume $n \in S$. Take any $m < n+1$. Then $m < n$ or $m = n$. Take the latter, we have that $F(n+1) = C(\{x \in B : x \prec F(n)\})$, since we have shown every $F(n) \in B$. But from this it is clear that $F(n+1) \in \{x \in B : x < F(n)\}$ and $F(n+1) \prec F(n) = F(m)$. Now take the former case, where $m < n$. By the inductive hypothesis, $F(n) \prec F(m)$. But this implies $F(n+1) \prec F(m)$, since we have just seen that $F(n+1) \prec F(n)$. Thus, in either case, $F(n+1) < F(m)$ so $n+1 \in S$. By induction $S = \omega$. Thus, F defines a strictly $\prec$-decreasing ω-sequence contained in A.

□

Proof of Theorem 1.

Proof. To see that $f[A_0, \dots, A_{n-1}]$ is well ordered, assume it is not. We have that $<$ is a linear order when restricted to $f[A_0, \dots, A_{n-1}]$. So there exists a strictly decreasing ω-sequence $(f(a_{0,i}, ..., a_{n-1,i}))_{i\in\omega}$ of elements of $f[A_0, \dots, A_{n-1}]$, i.e. the sequence satisfies $f(a_{0,j}, ..., a_{n-1,j}) < f(a_{0,i}, ..., a_{n-1,i})$ when $i < j$. Now we look to define an n-coloring on 2-subsets of the domain of this sequence, i.e. ω. Before doing so, note we have that, for every $i, j \in \omega$ with $i < j$, $f(a_{0,j}, ..., a_{n-1,j}) < f(a_{0,i}, ..., a_{n-1,i})$, which implies that $f(a_{0,i}, ..., a_{n-1,i}) \nleq f(a_{0,j}, ..., a_{n-1,j})$. Since f is weak order preserving, we have that $a_{0,i} \nleq a_{0,j}$, or $a_{1,i} \nleq a_{1,j}, \dots,$ or $a_{n-1,i} \nleq a_{n-1,j}$. So, for all $i < j$ with $i, j \in \omega$, there exists some $p \in n$ such that $a_{p,i} \nleq a_{p,j}$, i.e. $a_{p,j} < a_{p,i}$.

Now, take any $\{i, j\} \in [\omega]^2$. We may assume, w.l.o.g., that $i < j$. Then, let $\{i, j\} \mapsto p$ where p is the least element of n such that $a_{p,j} < a_{p,i}$. This map is well defined since such a p is, as seen above, guaranteed to exist and the "least" clause guarantees uniqueness and is ensured by n's being well ordered. Thus, this defines a coloring function $g : [\omega]^2 \to n$. But, by Ramsey's theorem, $\omega \to (\omega)^2_n$. Thus, there is some $p \in n$ and $\Gamma \subseteq \omega$ with $|\Gamma| = \omega$ such that, for each $\{i, j\} \in [\Gamma]^2$, we have $g(\{i, j\}) = p$.

Now we look to show this implies that A_p contains a strictly decreasing ω-sequence. To do this, consider the set $\{a_{p,i} \in A_p : i \in \Gamma\}$. Since $<$ restricted to Γ is a well order and $|\Gamma| = \omega$, $(\Gamma, <)$ is isomorphic to $(\omega, <)$. Let $\phi : \omega \to \Gamma$ be the unique isomorphism. This allow us to define a sequence $(a_{p,\phi(i)})_{i\in\omega}$. Now we must verify that $(a_{p,\phi(i)})_{i\in\omega}$ is strictly decreasing. Take any $i, j \in \omega$ such that $i < j$. Then $\phi(i) < \phi(j)$. Since $\phi(i), \phi(j) \in \Gamma$, it follows that $g(\{\phi(i), \phi(j)\}) = p$. Further, as $\phi(i) < \phi(j)$, we have $a_{p,\phi(j)} < a_{p,\phi(i)}$, by the definition of g. Thus the ω-sequence $(a_{p,\phi(i)})_{i\in\omega}$ is strictly decreasing and contained in A_p. Thus, A_p is not well ordered, a contradiction.

So, we may conclude that $f[A_0, \ldots, A_{n-1}]$ is well ordered, demonstrating that, if $A_0, \ldots, A_{n-1}$ are well-ordered subsets of $\mathbb{R}$, then the subset

$$f[A_0, \ldots, A_{n-1}] = \{f(a_0, \ldots, a_{n-1}) : a_0 \in A_0, \ldots, a_{n-1} \in A_{n-1}\}$$

of $\mathbb{R}$ is also well-ordered.

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