Set Theory (MATH 6730)

Infinite Combinatorics

We will discuss results in two topics: Δ-systems and partition calculus for infinite sets (= infinitary Ramsey theory).

1. Δ-Systems

Definition 1.1. A Δ-*system* (or *sunflower*) is a family $\mathcal{A}$ of sets with the property that there is a set r such that $A \cap B = r$ for any two distinct $A, B \in \mathcal{A}$. The set r is called the *root* or *kernel* of $\mathcal{A}$.

Theorem 1.2. (General Δ-System Theorem)
Let κ and λ be cardinals such that $\omega \leq \kappa < \lambda$, λ is regular, and $\left|[\alpha]^{<\kappa}\right| < \lambda$ for all $\alpha < \lambda$. If $\mathcal{A}$ is a collection of sets such that $|A| < \kappa$ for all $A \in \mathcal{A}$ and $|\mathcal{A}| \geq \lambda$, then there exists a Δ-system $\mathcal{B} \subseteq \mathcal{A}$ with $|\mathcal{B}| = \lambda$.

Proof. Replacing $\mathcal{A}$ by a subset of size λ, we may assume that $|\mathcal{A}| = \lambda$. Let $\lambda \to \mathcal{A}$, $\alpha \mapsto A_\alpha$ be a bijection. Now let $\mu := \kappa$ if κ is regular, and let $\mu := \kappa^+$ if κ is singular. Then:

- μ is a regular cardinal with $\mu < \lambda$.
  For singular κ use the assumptions to show that $\mu = \kappa^+ \leq \kappa^{\mathrm{cf}(\kappa)} < \lambda$.

- $S := \{\alpha < \lambda : \alpha \text{ is a limit ordinal and } \mathrm{cf}(\alpha) = \mu\}$ is a stationary subset of λ.

- There exists a one-to-one function $f\colon \bigcup \mathcal{A} \to \lambda$. Let $b_\alpha := f[A_\alpha]$ for each $\alpha < \lambda$.

- There exists a function $g\colon S \to \lambda$ such that $\bigcup(b_\alpha \cap \alpha) < g(\alpha) < \alpha$ for all $\alpha \in S$.

- By Fodor's Lemma[1], there exists $\beta < \lambda$ such that $S' := g^{-1}[\{\beta\}]\,(\subseteq S)$ is stationary in λ.
- $|S'| = \lambda$, and for all $\alpha \in S'$ we have that $b_\alpha \cap \alpha \in [\beta]^{<\kappa}$.

- The function $S' \to [\beta]^{<\kappa}$, $\alpha \mapsto b_\alpha \cap \alpha$ has a kernel class of size λ; that is, there exist $S'' \subseteq S'$ and $B \in [\beta]^{<\kappa}$ such that $|S''| = \lambda$ and $b_\alpha \cap \alpha = B$ for all $\alpha \in S''$.

- Now show that there exists a sequence $\langle \alpha_\xi : \xi < \lambda \rangle$ of elements of S'' such that
  (i) $\alpha_\xi > \alpha_\eta$ for all $\eta < \xi$, and
  (ii) $\alpha_\xi > \delta$ for every $\delta \in \bigcup_{\eta<\xi} b_{\alpha_\eta}$.

- Let $\mathcal{B} = \{A_{\alpha_\xi} : \xi < \lambda\}\,(\subseteq \mathcal{A})$ and let $r = f^{-1}[B]$. Then
  - $|\mathcal{B}| = \lambda$, and
  - $A_{\alpha_\xi} \cap A_{\alpha_\eta} = r$ for any $\eta < \xi < \lambda$, so $\mathcal{B}$ is a Δ-system.

$\square$

[1]See Theorem 20 on the handout "Clubs and Stationary Sets".

Theorem 1.2. (General Δ-System Theorem)
Let κ and λ be cardinals such that $\omega \leq \kappa < \lambda$, λ is regular, and $\left|[\alpha]^{<\kappa}\right| < \lambda$ for all $\alpha < \lambda$. If $\mathcal{A}$ is a collection of sets such that $|A| < \kappa$ for all $A \in \mathcal{A}$ and $|\mathcal{A}| \geq \lambda$, then there exists a Δ-system $\mathcal{B} \subseteq \mathcal{A}$ with $|\mathcal{B}| = \lambda$.

Corollary 1.3. (Δ-System Lemma or Δ-System Theorem)
Let λ be an uncountable regular cardinal. If $\mathcal{A}$ is a collection of finite sets with $|\mathcal{A}| \geq \lambda$, then there exists a Δ-system $\mathcal{B} \subseteq \mathcal{A}$ with $|\mathcal{B}| = \lambda$.

Proof. This is the special case $\kappa = \omega$ of Theorem 1.2. The hypotheses of the theorem are satisfied, because for every $\alpha < \lambda$ we have that

$$\left|[\alpha]^{<\omega}\right| \leq \sum_{n\in\omega} |\alpha|^n \leq \max(|\alpha|, \omega) < \lambda.$$

□

Corollary 1.4. *If CH holds and $\mathcal{A}$ is a collection of countable sets with $|\mathcal{A}| \geq \omega_2$, then there exists a Δ-system $\mathcal{B} \subseteq \mathcal{A}$ with $|\mathcal{B}| = \omega_2$.*

Proof. This is the special case $\kappa = \omega_1$, $\lambda = \omega_2$ of Theorem 1.2, assuming CH. The hypotheses of the theorem are satisfied, because for every $\alpha < \omega_2$, CH implies that

$$\left|[\alpha]^{<\omega_1}\right| \leq \omega_1^\omega = (2^\omega)^\omega = 2^\omega = \omega_1 < \omega_2.$$

□

Definition 1.5. An *indexed* Δ*-system* is a system $\langle A_i : i \in I\rangle$ of sets such that there is a set r (called the *root* or *kernel*) such that $A_i \cap A_j = r$ for any two distinct $i, j \in I$.

Notice that some (possibly all) of the sets A_i $(i \in I)$ may be equal.

Corollary 1.6. (Indexed Δ-System Theorem)
Let κ, λ be cardinals such that $\omega \leq \kappa < \lambda$, λ is regular, and $\left|[\alpha]^{<\kappa}\right| < \lambda$ for all $\alpha < \lambda$. If $\langle A_i : i \in I\rangle$ is a system of sets such that $|A_i| < \kappa$ for all $i \in I$ and $|I| \geq \lambda$, then there exists $J \subseteq I$ with $|J| = \lambda$ such that $\langle A_i : i \in J\rangle$ is an indexed Δ-system.

Proof. Consider the equivalence relation $\equiv$ on I defined by $i \equiv j$ iff $A_i = A_j$. If some $\equiv$-class has size λ, choose that to be J. Otherwise:

- $\equiv$ has $\geq \lambda$ equivalence classes.
- By AC, there exists $K \subseteq I$ such that K has exactly one element from each $\equiv$-class.
- Apply Theorem 1.2 to the family $\mathcal{A} = \{A_k : k \in K\}$.

□

2. Partition Calculus

Notation 2.1. Let ρ, σ, κ be cardinals with $\rho \neq 0$, and let $\langle \lambda_\alpha : \alpha < \rho \rangle$ be a sequence of cardinals such that $1 \leq \sigma \leq \lambda_\alpha \leq \kappa$ for all $\alpha < \rho$. We write

$$\kappa \to (\langle \lambda_\alpha : \alpha < \rho \rangle)^\sigma$$

to denote the following statement:

> *For every* $f \colon [\kappa]^\sigma \to \rho$ *there exist* $\alpha < \rho$ *and* $\Gamma \in [\kappa]^{\lambda_\alpha}$ *such that* $f\big[[\Gamma]^\sigma\big] = \{\alpha\}$.

Equivalently:

> *For every coloring of the* σ*-element subsets of* κ *by* ρ *colors, there exist* $\alpha < \rho$ *and a* λ_α*-element subset* Γ *of* κ *such that every* σ*-element subset of* Γ *has color* α.

If $\rho = r \in \omega$, we will write $\kappa \to (\lambda_0, \ldots, \lambda_{r-1})^\sigma$ instead of $\kappa \to (\langle \lambda_0, \ldots, \lambda_{r-1} \rangle)^\sigma$.

In the special case when $\lambda_\alpha = \lambda$ for all $\alpha < \rho$, we abbreviate $\kappa \to (\langle \lambda : \alpha < \rho \rangle)^\sigma$ by

$$\kappa \to (\lambda)^\sigma_\rho.$$

Thus, $\kappa \to (\lambda)^\sigma_\rho$ means that

> *For every* $f \colon [\kappa]^\sigma \to \rho$ *there exists* $\Gamma \in [\kappa]^\lambda$ *such that* $\big|f\big[[\Gamma]^\sigma\big]\big| = 1$.

Equivalently:

> *For every coloring of the* σ*-element subsets of* κ *by* ρ *colors, there exists a* λ*-element subset* Γ *of* κ *such that every* σ*-element subset of* Γ *has the same color.*

We say in this situation that Γ *is monochromatic* or *homogeneous*.

Facts 2.2. *Under the same assumptions as above, we have the following:*

(i) $\kappa \to (\langle \lambda_\alpha : \alpha < \rho \rangle)^\sigma$ *holds trivially for* $\rho = 1$.

(ii) $\kappa \to (\langle \lambda_\alpha : \alpha < \rho \rangle)^\sigma$ *holds for* $\sigma = 1$

- *if* $\kappa > \sum_{\alpha<\rho} \mu_\alpha$ *for all sequences* $\langle \mu_\alpha : \alpha < \rho \rangle$ *of cardinals such that* $\mu_\alpha < \lambda_\alpha$ *for each* $\alpha < \rho$;
- *in particular, if* ρ *is finite and* κ *is infinite.*

(iii) $\kappa \to (\langle \lambda_\alpha : \alpha < \rho \rangle)^\sigma$ *implies* $\kappa' \to (\langle \lambda'_\alpha : \alpha < \rho \rangle)^\sigma$, *if* $\kappa \leq \kappa'$ *and* $\lambda_\alpha \geq \lambda'_\alpha$ *for all* $\alpha < \rho$.

(iv) *For* $2 \leq r \in \omega$, *if* $\kappa \to (\lambda_0, \ldots, \lambda_{r-2}, \mu)^\sigma$ *and* $\mu \to (\lambda_{r-1}, \lambda_r)^\sigma$, *then*

$$\kappa \to (\lambda_0, \ldots, \lambda_{r-1}, \lambda_r)^\sigma.$$

Theorem 2.3. (Ramsey's Theorem) *For all nonzero natural numbers n and r,*

$$\omega \to (\omega)^n_r.$$

Sketch of Proof. By Facts 2.2(i) and (iv), our statement will follow by induction on r if we prove it for $r = 2$.

We will prove $\omega \to (\omega)^n_2$ by induction on n. The statement is clearly true for $n = 1$ (see Facts 2.2(ii)). Now we assume that $n \geq 1$ and $\omega \to (\omega)^n_2$ holds. To prove $\omega \to (\omega)^{n+1}_2$, let us consider an arbitrary (but fixed) coloring of the $(n+1)$-element subsets of ω by two colors (red and blue). For any nonempty set $S \subseteq \omega$, let $\min(S)$ denote the least element of S.

- Use the induction hypothesis to show that there exists a function

  $$H\colon \{(m, S) : m \in \omega,\ S \in [\omega \setminus m^+]^\omega\} \to [\omega \setminus m^+]^\omega$$

  such that the following conditions hold for all $(m, S) \in \mathrm{dmn}(H)$:
  - $H(m, S) \subseteq S$, and
  - every $(n+1)$-element set of the form $\{m\} \cup U$ with $U \in [H(m, S)]^n$ has the same color.

- Define a sequence $\langle \Gamma_k : k \in \omega \rangle$ of (infinite) subsets of ω by recursion on ω, using the notation $m_k := \min(\Gamma_k)$, as follows:

  $$\Gamma_0 = \omega \text{ (hence, } m_0 = 0\text{), and } \Gamma_{k+1} = H(m_k, \Gamma_k \setminus \{m_k\}) \text{ for all } k \in \omega.$$

  Then, for all $k \in \omega$,
  - $\Gamma_{k+1} \subset \Gamma_k$, $m_{k+1} > m_k$, and $m_j \in \Gamma_{k+1}$ for all $j \geq k + 1$;
  - every $(n + 1)$-element subset of ω of the form $\{m_k\} \cup U$ with $U \in [\Gamma_{k+1}]^n$ has the same color; we will refer to this color as 'the shade of m_k'.

- $\{m_k : k \in \omega\}$ has an infinite subset $\Gamma := \{m_k : k \in I\}$ whose members all have the same shade, say blue. Then every $(n + 1)$-element subset of Γ is blue. $\square$

The finite version of Ramsey's Theorem stated below can be deduced from Theorem 2.3.[2]

Corollary 2.4. (Ramsey's Theorem, finite version) *For all nonzero natural numbers* n, r *and* $\ell_0, \ldots, \ell_{r-1}$ *such that* $n \leq \ell_0, \ldots, \ell_{r-1}$, *there exists a natural number* $k \geq \ell_0, \ldots, \ell_{r-1}$ *such that*

$$k \to (\ell_0, \ldots, \ell_{r-1})^n.$$

Ramsey's Theorem implies the following fundamental properties of infinite partial orders and infinite linear orders.

Corollary 2.5. *If* $(P, <)$ *is an infinite partial order, then either* P *has an infinite antichain*[3] *or* P *has a subset order isomorphic to* $(\omega, <)$ *or* $(\omega, >)$.

Proof. Let $(P, \prec)$ be a well-ordering of P. Now we apply the consequence $|P| \to (\omega)^2_3$ of Ramsey's Theorem (i.e., a combination of Theorem 2.3 for $r = 3$ and Facts 2.2(iii)) to the coloring of the 2-element subsets of P defined as follows: $\{p, q\}$ with $p \prec q$ is red if $p < q$, blue if $p > q$, and yellow if p, q are incomparable. $\square$

Corollary 2.6. *If* $(P, <)$ *is an infinite linear order, then* P *has a subset order isomorphic to* $(\omega, <)$ *or* $(\omega, >)$.

[2]See pp. 270–271 of *Lectures on Set Theory* by J. Donald Monk.

[3]An infinite set in which any two distinct elements are incomparable.

Next, we want to show that the analogue of Ramsey's theorem fails for infinite successor cardinals (in place of ω), by showing that the analog of Corollary 2.6 fails for infinite successor cardinals.

Definition 2.7. Let κ be an infinite cardinal. The *lexicographic ordering of* ${}^{\kappa}2$ is defined as follows: for distinct elements $f = \langle f_\alpha : \alpha < \kappa \rangle$ and $g = \langle g_\alpha : \alpha < \kappa \rangle$ of ${}^{\kappa}2$, let

$$f < g \quad \Leftrightarrow \quad \text{for the least } \alpha < \kappa \text{ such that } f_\alpha \neq g_\alpha \text{ we have } f_\alpha = 0 < 1 = g_\alpha.$$

Theorem 2.8. *Let κ be an infinite cardinal.*

(i) *The linear order $({}^{\kappa}2, <)$, where $<$ is the lexicographic ordering of ${}^{\kappa}2$, has no subset order isomorphic to $(\kappa^+, <)$ or $(\kappa^+, >)$.*

(ii) $2^\kappa \not\to (\kappa^+)^2_2$.

Idea of Proof. (i) We proceed by contradiction. Assume there is a strictly increasing sequence $\langle f^{(\alpha)} : \alpha < \kappa^+ \rangle$ of elements of $({}^{\kappa}2, <)$. (The proof is similar if $({}^{\kappa}2, <)$ contains a strictly decreasing sequence $\langle f^{(\alpha)} : \alpha < \kappa^+ \rangle$ of elements.)

Claim 2.9. *If $\gamma \leq \kappa$, $\Gamma \in [\kappa^+]^{\kappa^+}$, and $f^{(\alpha)}\restriction\gamma < f^{(\beta)}\restriction\gamma$ for all $\alpha < \beta$ in Γ, then there exist $\delta < \gamma$ and $\Delta \in [\Gamma]^{\kappa^+}$ such that $f^{(\alpha)}\restriction\delta < f^{(\beta)}\restriction\delta$ for all $\alpha < \beta$ in Δ.*

This yields a strictly decreasing ω-sequence $\kappa = \gamma_0 > \gamma_1 > \cdots > \gamma_i > \gamma_{i+1} > \ldots$ of ordinals, which is impossible.

(ii) Suppose (ii) fails, that is, $2^\kappa \to (\kappa^+)^2_2$. Then the same argument as in the proof of Corollary 2.5 — except that we don't need the color yellow — shows that (i) fails. $\square$

Corollary 2.10. *If κ is an infinite cardinal, then $\kappa^+ \not\to (\kappa^+)^2_2$.*

This shows that Ramsey's theorem $\omega \to (\omega)^n_r$ does not generalize from ω to infinite successor cardinals κ^+. In fact, the statement *"There exists an uncountable cardinal λ such that $\lambda \to (\lambda)^2_2$."* cannot be proved in ZFC. Such a cardinal λ is called *weakly compact*. It is not hard to show that a weakly compact cardinal λ must be *(strongly) inaccessible*, i.e., it must be an uncountable regular cardinal which satisfies $2^\mu < \lambda$ for all cardinals $\mu < \lambda$. (The last property follows from Theorem 2.8(ii) and Facts 2.2(iii).)

Next we will discuss two less straightforward generalizations of Ramsey's Theorem. The first one is to be compared to the negative result in Corollary 2.10.

Theorem 2.11. (Dushnik–Miller Theorem)[4]
For every infinite regular cardinal κ, we have that $\kappa \to (\kappa, \omega)^2$.

Proof. [Proof will be presented in class.] $\square$

[4]The theorem is true for singular cardinals κ as well, but the proof is more difficult.

Another generalization of Ramsey's Theorem, to be compared to the negative result in Theorem 2.8(ii), is the following.

Theorem 2.12. (Erdős–Rado Theorem for $n = 2$)
For every infinite cardinal κ we have that $(2^\kappa)^+ \to (\kappa^+)^2_\kappa$.

This is the special case $n = 2$ of Theorem 2.14 below.

Definition 2.13. For every infinite cardinal κ, the ordinal class function $\beth(\kappa)$ assigning to every $\alpha \in \mathbf{On}$ a cardinal $\beth_\alpha(\kappa)$ is defined by recursion as follows: $\beth_0(\kappa) = \kappa$, and for all $\alpha \in \mathbf{On}$,

$$\beth_\alpha(\kappa) = 2^{\beth_\beta(\kappa)} \text{ if } \alpha = \beta + 1, \quad \text{and} \quad \beth_\alpha(\kappa) = \bigcup_{\beta<\alpha} \beth_\beta(\kappa) \text{ if } \alpha \text{ is limit.}$$

$\beth_\alpha$ stands for $\beth_\alpha(\omega) = \beth_\alpha(\aleph_0)$.[5]

Theorem 2.14. (Erdős–Rado Theorem)
For every infinite cardinal κ and for each nonzero natural number n,

$$(\dagger)_n \qquad \big(\beth_{n-1}(\kappa)\big)^+ \to (\kappa^+)^n_\kappa.$$

Proof. [Proof will be presented in class.] □

Theorem 2.8(ii) and the next theorem show that in Theorem 2.12, $(2^\kappa)^+$ cannot be replaced by 2^κ, even if the desired conclusion is significantly weakened.

Theorem 2.15. *For every infinite cardinal κ, $2^\kappa \not\to (3)^2_\kappa$.*

Proof. Consider the coloring of ${}^\kappa 2$ by κ colors defined as follows: for any two distinct elements $f = \langle f_\alpha : \alpha < \kappa \rangle$ and $g = \langle g_\alpha : \alpha < \kappa \rangle$ of ${}^\kappa 2$, color $\{f, g\}$ by the least ordinal $\alpha < \kappa$ such that $f_\alpha \neq g_\alpha$. Then there is no monochromatic 3-element subset in ${}^\kappa 2$. □

Theorem 2.16. $\omega \not\to (\omega)^\omega_2$.

Proof. Let $([\omega]^\omega, <)$ be a well-ordering, and define $F \colon [\omega]^\omega \to 2$ for each $X \in [\omega]^\omega$ by

$$F(X) := \begin{cases} 0 & \text{if there exists } Y \in [X]^\omega \text{ such that } Y < X, \\ 1 & \text{otherwise.} \end{cases}$$

Claim. *$F{\restriction}[H]^\omega$ is not constant for any $H \in [\omega]^\omega$.*

- For a contradiction, assume $F{\restriction}[H]^\omega$ is constant, and $\omega \to H$, $i \mapsto m_i$ is a bijection; so $H = \{m_i : i \in \omega\}$.
- Show that $F(X) = 1$ for all $X \in [H]^\omega$. (Hint: $F(X_0) = 1$ for the $<$-least element $X_0 \in [H]^\omega$).
- Consider the elements $Z_k = \{m_0, m_2, \dots, m_{2k}\} \cup \{m_{2i+1} : i \in \omega\}$ $(k \in \omega)$ of $[H]^\omega$, let Z_ℓ be $<$-least among them, and argue that $F(Z_{l+1}) = 0$, a contradiction.

□

[5] Alternative notation used in the literature for $\beth_n(\kappa)$ $(n \in \omega)$ include $\exp_n(\kappa)$ and 2^κ_n.