Set Theory (MATH 6730)

Forcing. The consistency of $\mathsf{ZFC} + \neg\mathrm{CH}$

Let M be a c.t.m. of ZFC. *Forcing* is a technique, developed by Paul Cohen (1963), to construct an extension M^* of M, which is another c.t.m. of ZFC, but also satisfies an additional property, e.g., $\neg$CH.

The idea — very roughly — is the following. To 'force' $\neg$CH to hold in M^*, we have to arrange that M^* contains an injection from the cardinal ω_2 in M^* to the cardinal 2^ω (or equivalently, to the set ${}^\omega 2$) in M^*.

- M may not contain such an injection $\omega_2 \to {}^\omega 2$ (for its own cardinals ω_2 and ω), but it contains finite 'partial descriptions' of such an injection, which can be viewed as finite partial functions $f \subseteq (\omega_2 \times \omega) \times 2$. So, let
$$P = \{f : f \text{ is a function with } \mathrm{dmn}(f) \subseteq \omega_2 \times \omega,\ \mathrm{rng}(f) \subseteq 2,\ |f| < \omega\}.$$
P is partially ordered by $\supseteq$, $P \ni \emptyset$, and $(P, \supseteq, \emptyset)$ is a member of the c.t.m. M; P will be called a *forcing order*.
- To be able to 'assemble' a total function $g\colon \omega_2 \times \omega \to 2$ from partial descriptions in such a way that g yields an injection $\omega_2 \to {}^\omega 2$, we extend M by a set $G \subseteq P$ (note: G may not be a member of the c.t.m. M), to get a new set $M^* = M[G]$ such that
  - $M \subseteq M[G]$, $G \in M[G]$ and $M[G]$ is a c.t.m. of ZFC;
  - $g = \bigcup G$ is a function in $M[G]$ which yields an injection $\omega_2 \to {}^\omega 2$ (for the cardinals ω_2 and ω in $M[G]$).

  A set $G \subseteq P$ used in the construction will be called a *filter P-generic over M*, and the new model $M[G]$ of ZFC will be called a *generic extension of M*.

After we

- define forcing orders P and filters P-generic over M in general, and study their existence and basic properties,

we have to face several major challenges in order to see that the idea sketched above works:

- Describe the members of a generic extension $M[G]$.
- Prove that $M[G]$ is a c.t.m. of ZFC.
- Prove that — under suitable assumptions (satisfied in the example above) — the construction $M \mapsto M[G]$ preserves cardinals. In particular, this means that ω, ω_1, and ω_2 are the same cardinals in $M[G]$ as in M; that is, the construction $M \mapsto M[G]$ does not introduce any 'unwanted' bijections $\omega \to \omega_1$ or $\omega_1 \to \omega_2$ in $M[G]$.

1. Forcing Orders and Generic Filters

Definition 1.1. A *forcing order* is a triple $\mathbb{P} = (P, \leq, 1)$ where P is a set, $\leq$ is a reflexive, transitive relation on P, and 1 is the largest element of P, i.e. $p \leq 1$ for all $p \in P$.

Note that antisymmetry is not required for $\leq$ in a forcing order $(P, \leq, 1)$, so $\leq$ may not be a partial order. Nevertheless, it is useful to think of the members of P as partial descriptions of a set we want to add to our model M, where $p \leq q$ means that p is a finer description than q.

Definition 1.2. Let $\mathbb{P} = (P, \leq, 1)$ be a forcing order.

- Two elements $p, q \in P$ are *compatible* (intuitively: have a common refinement) if there exists $r \in P$ such that $r \leq p, q$; otherwise, p, q are called *incompatible*, and we write $p \perp q$;
- A subset A of P is called an *antichain*[a] in $\mathbb{P}$ if any two distinct members of A are incompatible;
- A subset D of P is said to be *dense* in $\mathbb{P}$ if for every $p \in P$ there exists $d \in D$ such that $d \leq p$.

[a]This is yet another use of the word 'antichain'; it is different from — and not to be confused with — any of the two earlier meanings: (i) the usual order-theoretic meaning, see Definition 1 in the lecture notes 'Trees' and (ii) the use of the word as it applies to open subsets of a linear order, see Definition 16 in the lecture notes 'Trees'.

Definition 1.3. Let $\mathbb{P} = (P, \leq, 1)$ be a forcing order, and let M be a c.t.m. of ZFC.

- A *filter* on $\mathbb{P}$ is a subset G of P such that
  - for all $p, q \in G$ there exists $r \in G$ such that $r \leq p, q$, and
  - G is *up-closed*, that is, for all $g \in G$ and $p \in P$, if $g \leq p$, then $p \in G$.
- If $\mathbb{P} \in M$ and G is a subset of P (G is not necessarily a member of M), we say that G *is* $\mathbb{P}$*-generic over* M, provided
  - G is a filter on $\mathbb{P}$, and
  - for every dense subset D in $\mathbb{P}$ such that $D \in M$ we have that $G \cap D \neq \emptyset$.

Our first theorem shows that generic filters are usually not in the ground model.

Theorem 1.4. *Let M be a c.t.m. of* ZFC*, let $\mathbb{P} = (P, \leq, 1) \in M$ be a forcing order, and let G be a filter on $\mathbb{P}$ that is $\mathbb{P}$-generic over M. If $\mathbb{P}$ satisfies the condition*

$$(*) \qquad \text{for every } p \in P \text{ there exist } q, r \in P \text{ such that } q, r \leq p \text{ and } q \perp r,$$

then $G \notin M$.

Proof. Assuming $G \in M$, we get that

- $P \setminus G \in M$, because $P \in M$ and $x \setminus y$ is absolute for transitive (class) models of ZFC;
- $P \setminus G$ is dense in $\mathbb{P}$, by condition $(*)$,

which contradicts G being $\mathbb{P}$-generic. □

Theorem 1.5. *If M is a c.t.m. of* ZFC *and $\mathbb{P} = (P, \leq, 1) \in M$ is a forcing order, then for each $p \in P$ there exists a filter G on $\mathbb{P}$ such that G is $\mathbb{P}$-generic over M and $p \in G$.*

Proof. Let $\mathcal{D}$ be the set of all dense subsets D of P with $D \in M$. Since M is countable, there exists an onto function $\omega \to \mathcal{D}$, $n \mapsto D_n$. Now define a sequence $\langle q_n \rangle_{n\in\omega}$ of elements of P by recursion as follows: $q_0 = p$ and for all $n \in \omega$, given $q_n \in P$, let q_{n+1} be an element of D_n such that $q_{n+1} \leq q_n$. Then the set

$$G = \{r \in P : q_n \leq r \text{ for some } n \in \omega\}$$

- is a filter on $\mathbb{P}$, and
- is $\mathbb{P}$-generic over M.

Clearly, $p \in G$. □

2. The Generic Extension $M[G]$

We will define the set $M[G]$ by first defining P-names in M, and then constructing the elements of $M[G]$ by using P-names. P-names are certain sets in $\mathbf{V}$. We will obtain the class of all P-names via its characteristic class function, which is defined in the next theorem.

Theorem 2.1. *For any set P there exists a unique class function $\mathbf{F} = \mathbf{F}_P \colon \mathbf{V} \to 2$ such that for any set τ,*

$$\mathbf{F}(\tau) = \begin{cases} 1 & \textit{if } \tau \textit{ is a relation and} \\ & \textit{for all } (\sigma, p) \in \tau \textit{ we have that } p \in P \textit{ and } \mathbf{F}(\sigma) = 1, \\ 0 & \textit{otherwise.} \end{cases}$$

Definition 2.2. For any sets P and τ, τ is called a *P-name* if $\mathbf{F}_P(\tau) = 1$.

Proof of Theorem 2.1. Consider the following class relation on $\mathbf{V}$:

$$\mathbf{R} = \{(\sigma, \tau) : (\sigma, p) \in \tau \text{ for some } p \in P\}.$$

- If $(\sigma, \tau) \in \mathbf{R}$, i.e., $(\sigma, p) \in \tau$ for some $p \in P$, then $\sigma \in \{\sigma\} \in \{\{\sigma\}, \{\sigma, p\}\} = (\sigma, p) \in \tau$, so $\operatorname{rank}(\sigma) < \operatorname{rank}(\tau)$.
- Hence, $\mathbf{R}$ is well-founded and set-like on $\mathbf{V}$.

Applying the Recursion Theorem to $\mathbf{V}$, $\mathbf{R}$, and the class function $\mathbf{G} \colon \mathbf{V} \times \mathbf{V} \to 2$ defined by

$$\mathbf{G}(\tau, f) = \begin{cases} 1 & \text{if } \tau \text{ is a relation with } \operatorname{rng}(\tau) \subseteq P,\ f \text{ is a function with domain } \operatorname{pred}_{\mathbf{V},\mathbf{R}}(\tau), \\ & \text{and } f(\sigma) = 1 \text{ for all } \sigma \in \operatorname{pred}_{\mathbf{V},\mathbf{R}}(\tau), \\ 0 & \text{otherwise,} \end{cases}$$

we obtain $\mathbf{F}$. □

Facts 2.3. *Let P and τ be sets.*

(i) *'τ is a P-name' is absolute for all transitive (class) models of* ZFC *(in $\mathcal{L}_P$).*
(ii) *τ is a P-name if and only if τ is a relation such that for all $(\sigma, p) \in \tau$ we have that σ is a P-name and $p \in P$.*

Proof. (i) follows from Theorem 3.19 in the lecture notes 'Models of Set Theory'.
(ii) is an immediate consequence of Theorem 2.1 and Definition 2.2. □

Notation 2.4. For any set P, the class of all P-names is denoted by $\mathbf{V}^P$. If M is a c.t.m. of ZFC and $P \in M$, then the set $M \cap \mathbf{V}^P$ of P-names in M is denoted by M^P.

Note that if M is a c.t.m. of ZFC and $P \in M$, then by Fact 2.3(i) we have that

$$M^P = \{\tau \in M : (\tau \text{ is a } P\text{-name})^M\}.$$

Theorem 2.5. *For any set* $G \subseteq P$ *there exists a unique class function* $\mathrm{val}(-, G) = \mathrm{val}_P(-, G)\colon \mathbf{V} \to \mathbf{V}$ *such that for any set* τ,

$$\mathrm{val}(\tau, G) = \mathrm{val}_P(\tau, G) = \{\mathrm{val}(\sigma, G) : (\sigma, p) \in \tau \text{ for some } p \in G\}.$$

Proof. Let $\mathbf{R}$ be the same class relation as in the proof of Theorem 2.1. Then, applying the Recursion Theorem to $\mathbf{V}$, $\mathbf{R}$, and the class function $\mathbf{G}\colon \mathbf{V} \times \mathbf{V} \to \mathbf{V}$ defined by

$$\mathbf{G}(\tau, f) = \begin{cases} \{f(\sigma) : (\sigma, p) \in \tau \text{ for some } p \in G\} & \text{if } f \text{ is a function with domain } \mathrm{pred}_{\mathbf{V},\mathbf{R}}(\tau), \\ 0 & \text{otherwise,} \end{cases}$$

we obtain val. $\square$

Definition and Notation 2.6. Let $G \subseteq P$ be sets, and let M be a c.t.m. of ZFC.

- For any set τ, we write τ_G in place of $\mathrm{val}(\tau, G)$.
- If $P \in M$, we define $M[G]$ to be the set $\{\tau_G : \tau \in M^P\}$.

Facts 2.7. (i) $\mathrm{val}\,(= \mathrm{val}_P)$ *is absolute for transitive (class) models of* ZFC *(in* $\mathcal{L}_P$*).*

(ii) *If* M *is a c.t.m. of* ZFC *and* $G \subseteq P \in M$, *then*

$$\tau_G = \{\sigma_G : \sigma \in M^P,\ (\sigma, p) \in \tau \text{ for some } p \in G\} \qquad \text{for all } \tau \in M^P.$$

Theorem 2.8. *If M is a c.t.m. of* ZFC, $\mathbb{P} = (P, \leq, 1) \in M$ *is a forcing order, and G is a nonempty filter on $\mathbb{P}$, then $M[G]$ is a countable transitive set such that $M \subseteq M[G]$ and $G \in M[G]$; moreover, $M[G] \subseteq N$ for every c.t.m. N of* ZFC *such that $M \subseteq N$ and $G \in N$.*

Proof. (1) *$M[G]$ is a countable set:* $M[G]$ is a set by the Replacement Axiom, and it is countable, because M^P ($\subseteq M$) is.

(2) *$M[G]$ is transitive:* Let $x \in y \in M[G]$. Then $y = \tau_G$ for some $\tau \in M^P$. Thus $x \in \tau_G$, so $x = \sigma_G$ for some $\sigma \in M^P$, by Fact 2.7(ii). Hence, $x \in M[G]$.

(3) *$M \subseteq M[G]$:* Given $x \in M$ we have to find $\tau \in M^P$ such that $x = \tau_G$. In the next claim we construct a class function which, when restricted to M, produces such a τ from x.

Claim 2.9. *For each forcing order $\mathbb{P} = (P, \leq, 1)$ there exists a unique class function $\mathbf{F} = \mathbf{F}_{\mathbb{P}} \colon \mathbf{V} \to \mathbf{V}$ such that*

$$\mathbf{F}(x) = \{(\mathbf{F}(y), 1) : y \in x\} \qquad \textit{for all } x \in \mathbf{V}.$$

Proof of Claim 2.9. We obtain $\mathbf{F}$ by applying the Recursion Theorem to $\mathbf{V}$, the class relation $\mathbf{R} = \{(x, y) : x \in y\}$, and the class function $\mathbf{G} \colon \mathbf{V} \times \mathbf{V} \to \mathbf{V}$ defined by

$$\mathbf{G}(x, f) = \begin{cases} \{(f(y), 1) : y \in x\} & \text{if } f \text{ is a function with domain } x, \\ \emptyset & \text{otherwise.} \end{cases} \qquad \diamond$$

Notation 2.10. For any set x, we will write $\check{x}$ in place of $\mathbf{F}_{\mathbb{P}}(x)$ where $\mathbf{F}_{\mathbb{P}}$ is the class function from Claim 2.9. ($\check{x}$ depends on $\mathbb{P}$, but this dependence is suppressed in the notation.) Thus,

$$\check{x} = \{(\check{y}, 1) : y \in x\} \qquad \text{for every set } x.$$

Now $M \subseteq M[G]$ will follow if we show that for all $x \in M$ we have $\check{x} \in M^P$ and $\check{x}_G = x$. Assuming one of them fails for some $x \in \mathbf{V}$, consider an $\in$-minimal such x. Then

$$\check{x} = \{(\check{y}, 1) : y \in x\} \in M^P \quad \text{and}$$
$$\check{x}_G = \{\sigma_G : (\sigma, 1) \in \check{x}\} = \{\check{y}_G : (\check{y}, 1) \in \check{x}\} = \{y : y \in x\} = x,$$

which contradicts the choice of x.

(4) *$G \in M[G]$:* Let $\Gamma = \{(\check{p}, p) : p \in P\}$. It is clear from Fact 2.3(ii) that Γ is a P-name. Since M is a transitive model of ZFC and $P \in M$, we have that $\Gamma \in M$. Hence $\Gamma \in M \cap \mathbf{V}^P = M^P$. Since

$$\Gamma_G = \{\check{p}_G : (\check{p}, p) \in \Gamma \text{ for some } p \in G\} = \{\check{p}_G : p \in G\} = \{p : p \in G\} = G,$$

we get that $G \in M[G]$.

(5) *$M[G] \subseteq N$ for every c.t.m. N of* ZFC *such that $M \subseteq N$ and $G \in N$:* Suppose N is a c.t.m. of ZFC such that $M \subseteq N$ and $G \in N$. If $x \in M[G]$, say $x = \sigma_G$ with $\sigma \in M^P$, then $\sigma, G \in N$, so by Facts 2.7(i), $x = \sigma_G = \mathrm{val}(\sigma, G) \in N$. $\square$

Theorem 2.11. *If M is a c.t.m. of* ZFC*, $\mathbb{P} = (P, \leq, 1) \in M$ is a forcing order, and G is a nonempty filter on $\mathbb{P}$, then*

(i) $\operatorname{rank}(\tau_G) \leq \operatorname{rank}(\tau)$ *for all $\tau \in M^P$;*
(ii) *M and $M[G]$ have the same ordinals.*

Proof. (i) Assuming this fails for some $\tau \in M^P$, consider such a τ of minimal rank. Using Theorem 1.6(iv) in the lecture notes 'Models of Set Theory' and Facts 2.7(ii), we get that

$$\operatorname{rank}(\tau_G) = \bigcup\{\operatorname{rank}(\sigma_G) + 1 : \sigma \in M^P,\ (\sigma, p) \in \tau \text{ for some } p \in G\}$$

where for each σ on the right hand side we have that

- $\operatorname{rank}(\sigma) < \operatorname{rank}(\tau)$ (see the proof of Theorem 2.1), and
- $\operatorname{rank}(\sigma_G) \leq \operatorname{rank}(\sigma)$, by the choice of τ.

This implies that $\operatorname{rank}(\tau_G) \leq \operatorname{rank}(\tau)$, contrary to the choice of x.

(ii) Since $M \subseteq M[G]$ and ordinals are absolute for transitive classes, we get that every ordinal in M is an ordinal in $M[G]$. Conversely, let α be an ordinal in $M[G]$. Then $\alpha = \tau_G$ for some $\tau \in M^P$. Since rank is absolute for transitive (class) models of ZFC, we get that $\operatorname{rank}(\tau) \in M$. So, by (i), $\alpha \overset{!}{=} \operatorname{rank}(\alpha) = \operatorname{rank}(\tau_G) \leq \operatorname{rank}(\tau)$, where $\overset{!}{=}$ holds by Theorem 1.6(v) in the lecture notes 'Models of Set Theory'. Thus, $\alpha \in M$. $\square$

3. Forcing: The Relations $\Vdash$ and $\Vdash^*$

Now we introduce the main idea of forcing.

Definition 3.1. Let M be a c.t.m. of ZFC, and let $\mathbb{P} = (P, \leq, 1) \in M$ be a forcing order. For each formula $\varphi(v_0, \ldots, v_{m-1})$ in the language of set theory, we define another formula

$$p \Vdash_{\mathbb{P},M} \varphi(\sigma_0, \ldots, \sigma_{m-1})$$
$$\text{[read: } p \textit{ forces } \varphi(\sigma_0, \ldots, \sigma_{m-1}) \textit{ with respect to } \mathbb{P} \textit{ and } M\text{]},$$

which states that

> $\mathbb{P}$ is a forcing order, $\mathbb{P} \in M$, $\sigma_0, \ldots, \sigma_{m-1} \in M^P$, $p \in P$, and for every filter $G \subseteq P$ which is $\mathbb{P}$-generic over M, if $p \in G$, then the formula $\varphi^{M[G]}(v_0, \ldots, v_{m-1})$ (= the relativization of φ to $M[G]$) holds for the elements $\sigma_{0G}, \ldots, \sigma_{(m-1)G}$.

Recall that, in general, G is not a member of M. Therefore, the definition above cannot be given in M. Our aim is to define another notion $\Vdash^*$ such that

- $\Vdash^*$ is equivalent to $\Vdash$, and
- $\Vdash^*$ can be defined in any c.t.m. M of ZFC.

This will be done by assigning to each formula $\varphi(v_0, \ldots, v_{m-1})$ and all choices of $\sigma_0, \ldots, \sigma_{m-1} \in M^P$ a 'truth value', which will be a set in M. Ordinary truth values, 0 and 1, are sets, namely $\emptyset$ and $\{\emptyset\}$, and the operations on truth values that correspond to disjunction, conjunction, and negation are union, intersection, and $^{\mathrm{c}}$ (complement with respect to $\{\emptyset\}$) on the set $\big\{\emptyset, \{\emptyset\}\big\} = \mathcal{P}(\{\emptyset\})$. The 'truth values' we will introduce here will be special kinds of sets that encode a given forcing order $\mathbb{P} \,(\in M)$ and its elements by sets in M. Moreover, we will assign these 'truth values' in such a way that computations with them will be similar to computations with ordinary truth values.

The introduction of these 'truth values' requires some preparation, and some basic notions and facts from topology. Throughout, "x contains y" will mean that "y is a subset of x".

Definition 3.2. Let P be a set, and let $\mathcal{O} \subseteq \mathcal{P}(P)$.

(i) $\mathcal{O}$ is a *topology* on P if $\emptyset, P \in \mathcal{O}$, $\mathcal{O}$ is closed under unions of arbitrary subfamilies of $\mathcal{O}$, and $\mathcal{O}$ is closed under intersections of finite subfamilies of $\mathcal{O}$.

Given a fixed topology $\mathcal{O}$ on P, and a subset $X \subseteq P$,

(ii) we say that X is *open* if $X \in \mathcal{O}$, and X *closed* if its complement $X^{\mathrm{c}} := P \setminus X$ with respect to P is open;
(iii) the *interior* of X, denoted by $\mathrm{int}(X)$, is the union of all open sets contained in X;
(iv) the *closure* of X, denoted by $\mathrm{cl}(X)$, is the intersection of all closed sets that contain X;
(v) X is called *regular open* if $X = \mathrm{int}\big(\mathrm{cl}(X)\big)$.

Notation 3.3. The set of all regular open subsets of P (with respect to the topology $\mathcal{O}$) will be denoted by $\mathrm{RO}_{\mathcal{O}}(P)$ or by $\mathrm{RO}(P)$ (if $\mathcal{O}$ is clear from the context).

It is not hard to check that

- the set $\mathrm{int}\big(\mathrm{cl}(Y)\big)$ is regular open, and hence belongs to $\mathrm{RO}(P)$, for every $Y \subseteq P$.

The following theorem is fairly straightforward to prove, and expresses that — although $\mathrm{RO}(P)$ is usually not closed under $\bigcup$ (union), $\bigcap$ (intersection, where $\bigcap\emptyset := P$), and ${}^{\mathrm{c}}$ (complement with respect to P), there exist natural operations $\bigvee$ (join, l.u.b.), $\bigwedge$ (meet, g.l.b.), and $'$ (complement) on $\mathrm{RO}(P)$ which obey many of the usual rules of computation valid for $\bigcup$, $\bigcap$, and ${}^{\mathrm{c}}$ on $\mathcal{P}(P)$.

Theorem 3.4. *Let P be a set, and let $\mathcal{O}$ be a topology on P.*

(i) *For arbitrary sets $X \in \mathrm{RO}(P)$ and $\mathcal{S} \subseteq \mathrm{RO}(P)$,*
- $\bigvee\mathcal{S} := \mathrm{int}\Big(\mathrm{cl}\Big(\bigcup\mathcal{S}\Big)\Big)$ *is the least regular open set (with respect to $\subseteq$) which contains every member of $\mathcal{S}$;*
- $\bigwedge\mathcal{S} := \bigvee\{Y : Y \in \mathrm{RO}(P),\ Y \subseteq \bigcap\mathcal{S}\}$ *is the largest regular open set (with respect to $\subseteq$) which is contained in every member of $\mathcal{S}$;*
  in particular, if $\mathcal{S}$ is finite, then $\bigwedge\mathcal{S} = \bigcap\mathcal{S}$;
- $X' := \mathrm{int}(P \setminus X)$ *is the largest regular open set (with respect to $\subseteq$) which satisfies $X \cap X' = \emptyset$.*

(ii) *The structure $\big(\mathrm{RO}(P), \vee, \wedge, ', \emptyset, P\big)$ (with binary $\vee$ and $\wedge$) is a* Boolean algebra[1]*; i.e.,*
- *both operations $\vee$ and $\wedge$ are commutative and associative;*
- *both operations $\vee$ and $\wedge$ are* idempotent, *i.e.*
  $X \vee X = X$ *and* $X \wedge X = X$ *for all* $X \in \mathrm{RO}(P)$*;*
- $(X \vee Y) \wedge X = X$ *and* $(X \wedge Y) \vee X = X$ *for all* $X, Y \in \mathrm{RO}(P)$*;*
- *each one of $\vee$ and $\wedge$ distributes over the other;*
- $X \vee \emptyset = X$, $X \wedge \emptyset = \emptyset$, $X \vee P = P$, $X \wedge P = X$ *for all* $X \in \mathrm{RO}(P)$*;*
- $(X')' = X$, $X \vee X' = P$, *and* $X \wedge X' = \emptyset$ *for all* $X \in \mathrm{RO}(P)$*;*
- *the* De Morgan laws *hold:*
  $(X \vee Y)' = X' \wedge Y'$ *and* $(X \wedge Y)' = X' \vee Y'$ *for all* $X, Y \in \mathrm{RO}(P)$*;*
- $X \vee Y = Y \Leftrightarrow X \subseteq Y \Leftrightarrow X \wedge Y = X \Leftrightarrow X \wedge Y' = \emptyset$ *for all* $X, Y \in \mathrm{RO}(P)$.

(iii) *In fact, $\big(\mathrm{RO}(P), \bigvee, \bigwedge, ', \emptyset, P\big)$ (with the operations $\bigvee$, $\bigwedge$ of arbitrary finite or infinite arities) is a* complete Boolean algebra[2]*. In addition to* (ii)*, we have that*
- $\wedge$ *distributes over $\bigvee$, and $\vee$ distributes over $\bigwedge$;*
- *the De Morgan laws hold for both $\bigvee$ and $\bigwedge$.*

As usual, if $\mathcal{S} \subseteq \mathrm{RO}(P)$ is an indexed set, say, $\mathcal{S} = \{S_i : i \in I\}$, then we may write $\bigvee_{i\in I} S_i$ instead of $\bigvee\mathcal{S}$ and $\bigwedge_{i\in I} S_i$ instead of $\bigwedge\mathcal{S}$.

---

[1] For the definition of a *Boolean algebra* and *complete Boolean algebra*, see Section 13 (pp. 144 and 148) of "Lectures notes on Set Theory" by Donald J. Monk. These structures are defined by some of the properties listed in Theorem 3.4, while the others on the list can be derived from the defining properties. Note, however, that the symbols for the Boolean algebra operations in "Lectures in Set Theory" differ from ours; namely, $+$, $\sum$, $\cdot$, $\prod$, and $-$ are used instead of our $\vee$, $\bigvee$, $\wedge$, $\bigwedge$, and $'$, respectively.

We will use Theorem 3.4 for the topologies $\mathcal{O}_{\mathbb{P}}$ on P that are induced by forcing orders $\mathbb{P} = (P, \leq, 1)$, as described in the next definition.

Definition and Notation 3.5. Let $\mathbb{P} = (P, \leq, 1)$ be a forcing order.

- We will call a subset X of P *down-closed* if for every $x \in X$ and $p \in P$ with $p \leq x$ we have that $p \in X$.
- The *topology* $\mathcal{O}_{\mathbb{P}}$ *induced by* $\mathbb{P}$ is the set of all down-closed subsets of P.
- For any $p \in P$, let $P \downarrow p = \{q \in P : q \leq p\}$.

It is easy to check that if $\mathbb{P} = (P, \leq, 1)$ is a forcing order, then

- $\mathcal{O}_{\mathbb{P}}$ is a topology on P, and $(P \downarrow p) \in \mathcal{O}_{\mathbb{P}}$ for all $p \in P$.

Definition 3.6. Let $\mathbb{P} = (P, \leq, 1)$ be a forcing order, and let $\mathrm{RO}(\mathbb{P}) := \mathrm{RO}_{\mathcal{O}_{\mathbb{P}}}(P)$ be the set of all regular open sets in P with respect to the topology $\mathcal{O}_{\mathbb{P}}$ induced by $\mathbb{P}$. We define a function $e\colon P \to \mathrm{RO}(\mathbb{P})$ by

$$e(p) := \mathrm{int}\big(\mathrm{cl}(P \downarrow p)\big) \qquad \text{for all } p \in P.$$

Clearly, under the assumptions of this definition,

- $e(p) \in \mathrm{RO}(\mathbb{P})$ for all $p \in P$, and $e(1) = P$.

In order to state further important properties of e we need the following definition (cf. Definition 1.2).

Definition 3.7. Let $\mathbb{P} = (P, \leq, 1)$ be a forcing order. For an element $p \in P$ and a subset $X \subseteq P$ we say that X is *dense below* p in $\mathbb{P}$ if for every $r \in P$ with $r \leq p$ there exists $x \in X$ such that $x \leq r$.

Theorem 3.8. *Let $\mathbb{P} = (P, \leq, 1)$ be a forcing order, and let $p, q \in P$ and $a, b \in \mathrm{RO}(\mathbb{P})$.*

(i) *$e[P]$ is dense in $\mathrm{RO}(\mathbb{P}) \setminus \{\emptyset\}$, i.e., for any nonempty set $Y \in \mathrm{RO}(\mathbb{P})$ there exists $p \in P$ such that $e(p) \subseteq Y$.*

(ii) *$e(p) = \mathrm{int}\big(\mathrm{cl}(P \downarrow p)\big) = \{r \in P : \text{for all } u \in P \text{ with } u \leq r,\ u \text{ and } p \text{ are compatible}\}$. Hence, $p \perp q$ iff $e(p) \cap e(q) = \emptyset$.*

(iii) *The following conditions on p, q are equivalent:*
   (a) *$e(p) \subseteq e(q)$;*
   (b) *$\{r \in P : r \leq p, q\}$ is dense below p.*
   Consequently, $p \leq q$ implies that $e(p) \subseteq e(q)$, and $e(p) \subseteq e(q)$ implies that p, q are compatible.

Now we are ready to return to our main task: given a c.t.m. M of ZFC and a forcing order $\mathbb{P} = (P, \leq, 1) \in M$, we want to define a notion $\Vdash^*$ — a notion equivalent to forcing ($\Vdash_{\mathbb{P},M}$) — which requires assigning 'truth values' (from M) to formulas $\varphi(v_0, \ldots, v_{m-1})$ and all choices of P-names $\sigma_0, \ldots, \sigma_{m-1}$ in M for the variables. First we will define the assignment of 'truth values' for all P-names in $\mathbf{V}$ so that the 'truth values' are members of $\mathrm{RO}(\mathbb{P})$, and then we will relativize to M. We start with atomic formulas: $v_0 = v_1$ and $v_0 \in v_1$.

Theorem 3.9. *Let $\mathbb{P} = (P, \leq, 1)$ be a forcing order, and let e be the function $P \to \mathrm{RO}(\mathbb{P})$ from Definition 3.6. There exist two class functions $\mathbf{V}^P \times \mathbf{V}^P \to \mathrm{RO}(\mathbb{P})$, denoted*

$$(*) \qquad (\sigma, \tau) \mapsto [\![\sigma = \tau]\!] \qquad \textit{and} \qquad (\sigma, \tau) \mapsto [\![\sigma \in \tau]\!],$$

such that the following hold for all $\sigma, \tau \in \mathbf{V}^P$:

$$(\dagger) \qquad [\![\sigma = \tau]\!] = \bigwedge_{(\xi,p)\in\tau} (e(p)' \vee [\![\xi \in \sigma]\!]) \wedge \bigwedge_{(\eta,q)\in\sigma} (e(q)' \vee [\![\eta \in \tau]\!]),$$

$$(\ddagger) \qquad [\![\sigma \in \tau]\!] = \bigvee_{(\xi,p)\in\tau} (e(p) \wedge [\![\sigma = \xi]\!]).$$

Reading $\vee$, $\bigvee$ as 'or', $\wedge$, $\bigwedge$ as 'and', and $'$ as 'not', we see that

- $[\![\sigma = \tau]\!]$ means, in a sense, that every element of τ is an element of σ and every element of σ is an element of τ. Similarly,
- $[\![\sigma \in \tau]\!]$ means, in a sense, that there is an element in τ that is equal to σ.

We will refer to $[\![\sigma = \tau]\!]$ and $[\![\sigma \in \tau]\!]$ as the *Boolean value* of $\sigma = \tau$ and $\sigma \in \tau$, respectively.

Idea of Proof of Theorem 3.9. The desired properties ($\dagger$)–($\ddagger$) of the two class functions in ($*$) show that each one of them has the property that its value at a pair (σ, τ) depends on how the other function evaluates for elements of σ and/or τ. Therefore, the two class functions in ($*$) have to be defined simultaneously by recursion. In other words, the class function that we have to define by recursion is a class function $\mathbf{F}\colon 2 \times \mathbf{V}^P \times \mathbf{V}^P \to \mathrm{RO}(\mathbb{P})$ such that the equalities ($\dagger$)–($\ddagger$) hold for all $\sigma, \tau \in \mathbf{V}^P$ if every occurrence of $[\![x = y]\!]$ is replaced by $\mathbf{F}(0, x, y)$ and every occurrence of $[\![x \in y]\!]$ is replaced by $\mathbf{F}(1, x, y)$.

For the construction one can use the class relation $\mathbf{R}$ on the class $\mathbf{A} = 2 \times \mathbf{V}^P \times \mathbf{V}^P$, defined for any $(\overline{\delta}, \overline{\sigma}, \overline{\tau}), (\delta, \sigma, \tau) \in \mathbf{A}$ by

$$(\overline{\delta}, \overline{\sigma}, \overline{\tau})\, \mathbf{R}\, (\delta, \sigma, \tau) \quad \text{iff} \quad \begin{cases} \overline{\delta} = 1,\ \delta = 0,\ \overline{\tau} = \sigma,\ \mathrm{rank}(\overline{\sigma}) < \mathrm{rank}(\tau), & \text{or} \\ \overline{\delta} = 1,\ \delta = 0,\ \overline{\tau} = \tau,\ \mathrm{rank}(\overline{\sigma}) < \mathrm{rank}(\sigma), & \text{or} \\ \overline{\delta} = 0,\ \delta = 1,\ \overline{\sigma} = \sigma,\ \mathrm{rank}(\overline{\tau}) < \mathrm{rank}(\tau). \end{cases}$$

and the class function $\mathbf{G}\colon \mathbf{A} \times \mathbf{V} \to \mathrm{RO}(\mathbb{P})$ defined for any $(\delta, \sigma, \tau) \in \mathbf{A}$ and any function $f\colon \mathrm{pred}_{\mathbf{A},\mathbf{R}}\big((\delta, \sigma, \tau)\big) \to \mathrm{RO}(\mathbb{P})$ by

$$\mathbf{G}(0, \sigma, \tau, f) = \bigwedge_{(\xi,p)\in\tau} \big(e(p)' \vee f(1, \xi, \sigma)\big) \wedge \bigwedge_{(\eta,q)\in\sigma} \big(e(q)' \vee f(1, \eta, \tau)\big) \qquad \text{if } \delta = 0,$$

$$\mathbf{G}(1, \sigma, \tau, f) = \bigvee_{(\xi,p)\in\tau} \big(e(p) \wedge f(0, \sigma, \xi)\big) \qquad \text{if } \delta = 1,$$

and for any other choices of $(\delta, \sigma, \tau, f) \in \mathbf{A} \times \mathbf{V}$ by $\mathbf{G}(\delta, \sigma, \tau, f) = \emptyset$.

It is not hard to check that $\mathbf{R}$ is well-founded and set-like on $\mathbf{A}$, and an application of the Recursion Theorem to $\mathbf{A}$, $\mathbf{R}$, and $\mathbf{G}$ yields the desired class function $\mathbf{F}$. $\square$

Definition 3.10. Let $\mathbb{P}$ be a forcing order. Having defined Boolean values for $\sigma = \tau$ and $\sigma \in \tau$ ($\sigma, \tau \in \mathbf{V}^P$), we now use recursion to extend the definition to arbitrary formulas and any P-names assigned to the variables as follows: for arbitrary formulas φ, ψ (with all free variables among $v_0, \ldots, v_{m-1}$) and for all $\sigma_0, \ldots, \sigma_{m-1} \in \mathbf{V}^P$,

$$\begin{aligned}
[\![\neg\varphi(\sigma_0, \ldots, \sigma_{m-1})]\!] &:= [\![\varphi(\sigma_0, \ldots, \sigma_{m-1})]\!]', \\
[\![\varphi(\sigma_0, \ldots, \sigma_{m-1}) \rightarrow \psi(\sigma_0, \ldots, \sigma_{m-1})]\!] &:= [\![\varphi(\sigma_0, \ldots, \sigma_{m-1})]\!]' \vee [\![\psi(\sigma_0, \ldots, \sigma_{m-1})]\!], \\
[\![\forall x\, \varphi(\sigma_0, \ldots, \sigma_{m-1}, x)]\!] &:= \bigwedge_{\tau \in \mathbf{V}^P} [\![\varphi(\sigma_0, \ldots, \sigma_{m-1}, \tau)]\!] \\
&= \bigwedge \{a \in \mathrm{RO}(\mathbb{P}) : \exists \tau \in \mathbf{V}^P\, a = [\![\varphi(\sigma_0, \ldots, \sigma_{m-1}, \tau)]\!]\}.
\end{aligned}$$

Fact 3.11. *Under the same assumptions as in the preceding definition, we have that*

$$\begin{aligned}
[\![\varphi(\sigma_0, \ldots, \sigma_{m-1}) \vee \psi(\sigma_0, \ldots, \sigma_{m-1})]\!] &= [\![\varphi(\sigma_0, \ldots, \sigma_{m-1})]\!] \vee [\![\psi(\sigma_0, \ldots, \sigma_{m-1})]\!], \\
[\![\varphi(\sigma_0, \ldots, \sigma_{m-1}) \wedge \psi(\sigma_0, \ldots, \sigma_{m-1})]\!] &= [\![\varphi(\sigma_0, \ldots, \sigma_{m-1})]\!] \wedge [\![\psi(\sigma_0, \ldots, \sigma_{m-1})]\!], \\
[\![\exists x\, \varphi(\sigma_0, \ldots, \sigma_{m-1}, x)]\!] &= \bigvee_{\tau \in \mathbf{V}^P} [\![\varphi(\sigma_0, \ldots, \sigma_{m-1}, \tau)]\!].
\end{aligned}$$

Definition 3.1. Let M be a c.t.m. of ZFC, and let $\mathbb{P} = (P, \leq, 1) \in M$ be a forcing order. For each formula $\varphi(v_0, \dots, v_{m-1})$ in the language of set theory, we define another formula

$$p \Vdash_{\mathbb{P},M} \varphi(\sigma_0, \dots, \sigma_{m-1})$$

[read: *p forces $\varphi(\sigma_0, \dots, \sigma_{m-1})$ with respect to $\mathbb{P}$ and M*],

which states that

> $\mathbb{P}$ is a forcing order, $\mathbb{P} \in M$, $\sigma_0, \dots, \sigma_{m-1} \in M^P$, $p \in P$, and for every filter $G \subseteq P$ which is $\mathbb{P}$-generic over M, if $p \in G$, then the formula $\varphi^{M[G]}(v_0, \dots, v_{m-1})$ (= the relativization of φ to $M[G]$) holds for the elements $\sigma_{0G}, \dots, \sigma_{(m-1)G}$.

Definition 3.12. Let $\mathbb{P}$ be a forcing order, and let $p \in P$. For arbitrary formula φ (with all free variables among $v_0, \dots, v_{m-1}$) and for all $\sigma_0, \dots, \sigma_{m-1} \in \mathbf{V}^P$, we define $p \Vdash^* \varphi(\sigma_0, \dots, \sigma_{m-1})$ to mean that $e(p) \subseteq [\![\varphi(\sigma_0, \dots, \sigma_{m-1})]\!]$.

The Forcing Theorem 3.13. *Let M be a c.t.m of* ZFC*, let $\mathbb{P} \in M$ be a forcing order, and let $G \subseteq P$ be a filter that is $\mathbb{P}$-generic over M. For any formula φ (with all free variables among $v_0, \dots, v_{m-1}$) and for any $\sigma_0, \dots, \sigma_{m-1} \in M^P$, the following conditions are equivalent:*

(a) *$\varphi(\sigma_{0G}, \dots, \sigma_{(m-1)G})$ holds in $M[G]$.*

(b) *There is a $p \in G$ such that $\big(p \Vdash^* \varphi(\sigma_0, \dots, \sigma_{m-1})\big)^M$.*

Remark 3.14. Let M, $\mathbb{P}$, φ, $\sigma_0, \dots, \sigma_{m-1}$ be as in the Forcing Theorem, and let $p \in P$. To relativize $p \Vdash^* \varphi(\sigma_0, \dots, \sigma_{m-1})$ to M, we have to relativize the equivalent (defining) condition $e(p) \subseteq [\![\varphi(\sigma_0, \dots, \sigma_{m-1})]\!]$ to M. This requires

- relativizing first the Boolean algebra $\mathrm{RO}(\mathbb{P})$ to M, to get an analogous Boolean algebra $\mathrm{RO}(\mathbb{P})^M$ in M, and then
- constructing the Boolean values $[\![\varphi(\sigma_0, \dots, \sigma_{m-1})]\!]^M$ in M, as in Theorem 3.9 and Definition 3.12, but using the Boolean algebra $\mathrm{RO}(\mathbb{P})^M$ instead of $\mathrm{RO}(\mathbb{P})$.

Note that — although $\mathbb{P} \in M$ — the Boolean algebras $\mathrm{RO}(\mathbb{P})$ and $\mathrm{RO}(\mathbb{P})^M$ are very different. In all interesting cases, the differences start already with the underlying topologies: $\mathcal{O}_{\mathbb{P}}$ in $\mathbf{V}$, and its relativization $\mathcal{O}_{\mathbb{P}}^M$ to M. E.g., if $G \subseteq P$ is a filter on $\mathbb{P}$ such that $G \notin M$, then $P \setminus G$ is down-closed, so $P \setminus G \in \mathcal{O}_{\mathbb{P}}$, but $P \setminus G \notin M$, so $P \setminus G \notin \mathcal{O}_{\mathbb{P}}^M$. Another fact: $\mathrm{RO}(\mathbb{P})^M$ and $\mathrm{RO}(\mathbb{P})$ are much different in that the complete Boolean algebra $\mathrm{RO}(\mathbb{P})$ is either finite or uncountable (as a set in $\mathbf{V}$), but $\mathrm{RO}(\mathbb{P})^M$ ($\subseteq M$) is countable (as a set in $\mathbf{V}$).

Warning: In what follows, we will omit the superscript ^{M} from $\mathrm{RO}(\mathbb{P})^M$ and its elements, but whenever we work with a condition $\big(p \Vdash^* \varphi(\sigma_0, \dots, \sigma_{m-1})\big)^M$, the corresponding computations will take place in $\mathrm{RO}(\mathbb{P})^M$.

Definition 3.1. Let M be a c.t.m. of ZFC, and let $\mathbb{P} = (P, \leq, 1) \in M$ be a forcing order. For each formula $\varphi(v_0, \ldots, v_{m-1})$ in the language of set theory, we define another formula

$$p \Vdash_{\mathbb{P},M} \varphi(\sigma_0, \ldots, \sigma_{m-1})$$

[read: *p forces* $\varphi(\sigma_0, \ldots, \sigma_{m-1})$ *with respect to* $\mathbb{P}$ *and* M],

which states that

> $\mathbb{P}$ is a forcing order, $\mathbb{P} \in M$, $\sigma_0, \ldots, \sigma_{m-1} \in M^P$, $p \in P$, and for every filter $G \subseteq P$ which is $\mathbb{P}$-generic over M, if $p \in G$, then the formula $\varphi^{M[G]}(v_0, \ldots, v_{m-1})$ (= the relativization of φ to $M[G]$) holds for the elements $\sigma_{0G}, \ldots, \sigma_{(m-1)G}$.

Theorem 3.8. *Let $\mathbb{P} = (P, \leq, 1)$ be a forcing order, and let $p, q \in P$ and $a, b \in \mathrm{RO}(\mathbb{P})$.*

(i) *$e[P]$ is dense in $\mathrm{RO}(\mathbb{P}) \setminus \{\emptyset\}$, i.e., for any nonempty set $Y \in \mathrm{RO}(\mathbb{P})$ there exists $p \in P$ such that $e(p) \subseteq Y$.*

(ii) *$e(p) = \mathrm{int}\big(\mathrm{cl}(P \downarrow p)\big) = \{r \in P :$ for all $u \in P$ with $u \leq r$, u and p are compatible$\}$. Hence, $p \perp q$ iff $e(p) \cap e(q) = \emptyset$.*

(iii) *The following conditions on p, q are equivalent:*
  (a) *$e(p) \subseteq e(q)$;*
  (b) *$\{r \in P : r \leq p, q\}$ is dense below p.*
Hence $p \leq q$ implies that $e(p) \subseteq e(q)$, and $e(p) \subseteq e(q)$ implies that p, q are compatible.

Definition 3.12. Let $\mathbb{P}$ be a forcing order, and let $p \in P$. For arbitrary formula φ (with all free variables among $v_0, \ldots, v_{m-1}$) and for all $\sigma_0, \ldots, \sigma_{m-1} \in \mathbf{V}^P$, we define $p \Vdash^* \varphi(\sigma_0, \ldots, \sigma_{m-1})$ to mean that $e(p) \subseteq [\![\varphi(\sigma_0, \ldots, \sigma_{m-1})]\!]$.

The Forcing Theorem 3.13. *Let M be a c.t.m of ZFC, let $\mathbb{P} \in M$ be a forcing order, and let $G \subseteq P$ be a filter that is $\mathbb{P}$-generic over M. For any formula φ (with all free variables among $v_0, \ldots, v_{m-1}$) and for any $\sigma_0, \ldots, \sigma_{m-1} \in M^P$, the following conditions are equivalent:*

(a) *$\varphi(\sigma_{0G}, \ldots, \sigma_{(m-1)G})$ holds in $M[G]$.*

(b) *There is a $p \in G$ such that $\big(p \Vdash^* \varphi(\sigma_0, \ldots, \sigma_{m-1})\big)^M$.*

Proof of the Forcing Theorem. Most work goes into proving the equivalence of (a)–(b) for atomic formulas. This is done by induction on the class relation $\mathbf{R}$ used in the simultaneous definitions of $[\![\sigma = \tau]\!]$ and $[\![\sigma \in \tau]\!]$ (by recursion on $\mathbf{R}$) in the proof of Theorem 3.9.

$\boxed{\text{(b)} \Rightarrow \text{(a) for } v_0 = v_1}$

Let $\sigma, \tau \in M^P$, and assume that there exists $p \in G$ such that $(p \Vdash^* \sigma = \tau)^M$, that is, such that <u>in M</u> we have

$$e(p) \subseteq [\![\sigma = \tau]\!] = \bigwedge_{(\xi,r)\in\tau} (e(r)' \vee [\![\xi \in \sigma]\!]) \wedge \bigwedge_{(\eta,q)\in\sigma} (e(q)' \vee [\![\eta \in \tau]\!]). \tag{1}$$

Our goal is to show that $\sigma_G = \tau_G$. By symmetry, it suffices to argue that $\sigma_G \subseteq \tau_G$.
Let $a \in \sigma_G = \{\eta_G : (\eta, q) \in \sigma \text{ for some } q \in G\}$; say $a = \eta_G$ with $(\eta, q) \in \sigma$, $q \in G$. Then:

1. $e(p) \subseteq e(q)' \vee [\![\eta \in \tau]\!]$, by (1).
2. $e(p) \wedge e(q) \subseteq [\![\eta \in \tau]\!]$, by 1 ($\wedge$ both sides with $e(q)$, and use the distributive law).
3. there exists $r \in G$ with $r \leq p, q$, since G is a filter on $\mathbb{P}$.
4. $e(r) \subseteq e(p) \wedge e(q) \subseteq [\![\eta \in \tau]\!]$, by 3 combined with Theorem 3.8(iii), and by 2.
5. $(r \Vdash^* \eta \in \tau)^M$, by the definition of $(\Vdash^*)^M$.
6. $a = \eta_G \in \tau_G$, by the induction hypothesis.

$\boxed{\text{(a)} \Rightarrow \text{(b) for } v_0 = v_1}$

Assume $\sigma_G = \tau_G$. Our goal is to show that there exists $p \in G$ such that $(p \Vdash^* \sigma = \tau)^M$.

Claim 3.15. *The following set is in M, and is dense in $\mathbb{P}$:*

$$D = \big\{p \in P : (p \Vdash^* \sigma = \tau)^M \ \textit{or}\ \exists(\xi, r) \in \tau \left(p \le r \wedge e(p) \subseteq [\![\xi \in \sigma]\!]'\right) \\ \textit{or}\ \exists(\eta, q) \in \sigma \left(p \le q \wedge e(p) \subseteq [\![\eta \in \tau]\!]'\right)\big\}.$$

Proof of Claim 3.15. $D \in M$ is clear by Cmpr. To prove the density of D, let $s \in P$. We want to find $p \in D$ such that $p \le s$. If $(s \Vdash^* \sigma = \tau)^M$, then $s \in D$ and we are done. So, assume that $(s \nVdash^* \sigma = \tau)^M$, that is, <u>in M</u>, we have that $e(s) \nsubseteq [\![\sigma = \tau]\!]$. Then:

1 $\emptyset \neq e(s) \wedge [\![\sigma = \tau]\!]' = e(s) \wedge \left(\bigvee_{(\xi,r)\in\tau}(e(r) \wedge [\![\xi \in \sigma]\!]') \vee \bigvee_{(\eta,q)\in\sigma}(e(q) \wedge [\![\eta \in \tau]\!]')\right)$.
2 $\emptyset \neq \bigvee_{(\xi,r)\in\tau}(e(s) \wedge e(r) \wedge [\![\xi \in \sigma]\!]') \vee \bigvee_{(\eta,q)\in\sigma}(e(s) \wedge e(q) \wedge [\![\eta \in \tau]\!]')$, by the distr. law.
3 One of the following holds:
   - there exists $(\xi, r) \in \tau$ such that $\emptyset \neq e(s) \wedge e(r) \wedge [\![\xi \in \sigma]\!]'$,
   - there exists $(\eta, q) \in \sigma$ such that $\emptyset \neq e(s) \wedge e(q) \wedge [\![\eta \in \tau]\!]'$,

   say the first.
4 There exists $t \in P$ with $e(t) \subseteq e(s) \wedge e(r) \wedge [\![\xi \in \sigma]\!]'$, because $e[P]$ is dense in $\mathrm{RO}(\mathbb{P}) \setminus \{\emptyset\}$ (Theorem 3.8(i)).
5 There exists $u \in P$ with $u \le t, r$, because $e(t) \subseteq e(r)$ implies t, r are compatible (Theorem 3.8(iii)).
6 There exists $p \in P$ with $p \le u, s$ similarly, because $e(u) \subseteq e(t) \subseteq e(s)$ holds by Theorem 3.8(iii) and 4.
7 Thus, $p \le r$ by 6, 5, and $e(p) \subseteq e(t) \subseteq [\![\xi \in \sigma]\!]'$ by $p \le t$ and 4.
8 Hence, $p \in D$ by 7, and $p \le s$ by 6. ◇

Since G is $\mathbb{P}$-generic over M, we get that $G \cap D \neq \emptyset$. Choose $p \in G \cap D$.

Claim 3.16. $(p \Vdash^* \sigma = \tau)^M$.

Proof of Claim 3.16. Suppose the claim is false. Since $p \in D$, we get:

1 One of the following holds:
   - there exists $(\xi, r) \in \tau$ such that $p \le r$ and $e(p) \subseteq [\![\xi \in \sigma]\!]'$,
   - there exists $(\eta, q) \in \sigma$ such that $p \le q$ and $e(p) \subseteq [\![\eta \in \tau]\!]'$,

   say the first.
2 $\emptyset \neq e(p) \nsubseteq [\![\xi \in \sigma]\!]$, so $(p \nVdash^* \xi \in \sigma)^M$.
3 $\xi_G \notin \sigma_G$, by the induction hypothesis.
4 $\xi_G \in \tau_G$, because $(\xi, r) \in \tau$ and $r \in G$ (as $G \ni p \le r$).
5 Thus, $\sigma_G \neq \tau_G$, contradicting our assumption. ◇

$\boxed{\text{(a)} \Leftrightarrow \text{(b) for } v_0 \in v_1}$

This proof uses similar techniques, but — unlike in the preceding case — the implication (b) $\Rightarrow$ (a) is the harder one.

For non-atomic formulas, we prove (a) $\Leftrightarrow$ (b) by induction on formulas. Let $\varphi = \varphi(v_0, \ldots, v_{m-1})$, and let $\sigma_0, \ldots, \sigma_{m-1} \in M^P$. We will use the abbreviations $\overline{\sigma}$ for $(\sigma_0, \ldots, \sigma_{m-1})$ and $\overline{\sigma}_G$ for $(\sigma_{0G}, \ldots, \sigma_{(m-1)G})$.

$\boxed{\text{(b)} \Rightarrow \text{(a) for } \neg\varphi}$

Assume $(p \Vdash^* \neg\varphi(\overline{\sigma}))^M$ holds for some $p \in G$, but $\neg\varphi(\overline{\sigma}_G)$ fails in $M[G]$, i.e., $\varphi(\overline{\sigma}_G)$ holds in $M[G]$.

① There is a $q \in G$ such that $(q \Vdash^* \varphi(\overline{\sigma}))^M$, by the hypothesis that (a) $\Leftrightarrow$ (b) holds for φ.
② There exists $r \in G$ with $r \leq p, q$, as G is a filter; so $e(r) \subseteq e(p), e(q)$.
③ $(r \Vdash^* \neg\varphi(\overline{\sigma}))^M$, because $(p \Vdash^* \neg\varphi(\overline{\sigma}))^M$ and $e(r) \subseteq e(p)$, and
$(r \Vdash^* \varphi(\overline{\sigma}))^M$, because $(q \Vdash^* \varphi(\overline{\sigma}))^M$ and $e(r) \subseteq e(q)$, contradiction.

$\boxed{\text{(a)} \Rightarrow \text{(b) for } \neg\varphi}$

Suppose $\neg\varphi(\overline{\sigma}_G)$ holds in $M[G]$.

Claim 3.17. *The set* $D = \left\{p \in P : \left(p \Vdash^* \varphi(\overline{\sigma})\right)^M \text{ or } \left(p \Vdash^* \neg\varphi(\overline{\sigma})\right)^M\right\} \in M$ *is dense in* $\mathbb{P}$.

Proof of Claim 3.17. Let $q \in P$ be arbitrary. Our goal is to find $p \in D$ such that $p \leq q$. If $\left(q \Vdash^* \varphi(\overline{\sigma})\right)^M$, then $q \in D$, so we are done. Assume therefore that $\left(q \nVdash^* \varphi(\overline{\sigma})\right)^M$. Then

① $e(q) \nsubseteq [\![\varphi(\overline{\sigma})]\!]$, so $e(q) \wedge [\![\varphi(\overline{\sigma})]\!]' \neq \emptyset$, as before.
② There exists $r \in P$ with $e(r) \subseteq e(q) \wedge [\![\varphi(\overline{\sigma})]\!]'$, by Theorem 3.8(i), as before.
③ r, q are compatible by Theorem 3.8(iii), as before, so there exists $p \in P$ with $p \leq r, q$.
④ $e(p) \subseteq e(r) \subseteq [\![\varphi(\overline{\sigma})]\!]' = [\![\neg\varphi(\overline{\sigma})]\!]$, therefore $\left(p \Vdash^* \neg\varphi(\overline{\sigma})\right)^M$.
⑤ So, $p \in D$ by ④, and $p \leq q$ by ③. ◇

$G \cap D \neq \emptyset$, because G is $\mathbb{P}$-generic over M. Choose $p \in G \cap D$. Were $\left(p \Vdash^* \varphi(\overline{\sigma})\right)^M$, our induction hypothesis would imply that $\varphi(\overline{\sigma}_G)$ holds in $M[G]$, which is impossible. Hence $p \in D$ yields that $\left(p \Vdash^* \neg\varphi(\overline{\sigma})\right)^M$.

$\boxed{\text{(a)} \Leftrightarrow \text{(b) for } \varphi \to \psi \text{ and } \forall v_i\, \varphi}$

Similar. $\square$

Now we are ready to prove that, for any c.t.m. M of ZFC, and for any forcing order $\mathbb{P} \in M$, the forcing notion $\Vdash_{\mathbb{P},M}$ (see Definition 3.1), which we will simply denote by $\Vdash$, and the forcing notion $\Vdash^*$ (see Definition 3.12), when relativized to M, are equivalent.

Corollary 3.18. *Let M be a c.t.m. of* ZFC*, let $\mathbb{P} = (P, \leq, 1) \in M$ be a forcing order, and let $p \in P$. For arbitrary formula φ and for any tuple $\overline{\tau}$ of P-names in M for the free variables of φ,*

$$p \Vdash \varphi(\overline{\tau}) \qquad \textit{iff} \qquad \big(p \Vdash^* \varphi(\overline{\tau})\big)^M.$$

Proof. $\Rightarrow$: Assume that $p \Vdash \varphi(\overline{\tau})$, but $\big(p \nVdash^* \varphi(\overline{\tau})\big)^M$. Then

1. $e(p) \nsubseteq [\![\varphi(\overline{\tau})]\!]$, so $e(p) \wedge [\![\varphi(\overline{\tau})]\!]' \neq \emptyset$.
2. There exists $q \in P$ with $e(q) \subseteq e(p) \wedge [\![\varphi(\overline{\tau})]\!]'$, by Theorem 3.8(i).
3. There exists $r \in P$ with $r \leq p, q$, because $e(q) \subseteq e(p)$ implies by Theorem 3.8(iii) that p, q are compatible.
4. $\big(r \Vdash^* \neg\varphi(\overline{\tau})\big)^M$, because $e(r) \subseteq e(q) \subseteq [\![\varphi(\overline{\tau})]\!]' = [\![\neg\varphi(\overline{\tau})]\!]$.
5. There exists a filter G on $\mathbb{P}$ such that $r \in G$ and G is $\mathbb{P}$-generic over M (Theorem 1.5).
6. $\neg\varphi^{M[G]}(\overline{\tau}_G)$ (i.e., $\neg\varphi(\overline{\tau}_G)$ holds in $M[G]$), by the Forcing Theorem.
7. $\varphi^{M[G]}(\overline{\tau}_G)$, because $p \Vdash \varphi(\overline{\tau})$ and $p \in G$ (as $G \ni r \leq p$); contradiction.

$\Leftarrow$: Assume that $\big(p \Vdash^* \varphi(\overline{\tau})\big)^M$. For any filter G on $\mathbb{P}$ such that $p \in G$ and G is $\mathbb{P}$-generic over M, the Forcing Theorem implies that $\varphi^{M[G]}(\overline{\tau}_G)$. This proves $p \Vdash \varphi(\overline{\tau})$. $\square$

Corollary 3.19. *Let M be a c.t.m. of* ZFC*, let $\mathbb{P} = (P, \leq, 1) \in M$ be a forcing order, and let $G \subseteq P$ be a filter that is $\mathbb{P}$-generic over M. For arbitrary formula φ and for any tuple $\overline{\tau}$ of P-names in M for the free variables of φ, the following conditions are equivalent:*

(a) $\varphi^{M[G]}(\overline{\tau}_G)$ *(i.e., $\varphi(\overline{\tau}_G)$ holds in $M[G]$).*

(b) *There exists $p \in G$ such that $p \Vdash \varphi(\overline{\tau})$.*

Proof. Combine the Forcing Theorem with Corollary 3.18. $\square$

4. Generic extensions $M[G]$ are models of ZFC

Definition 4.1. Let $\mathbb{P} = (P, \leq, 1)$ be a forcing order. For any P-names σ and τ, let

$$\text{u.p.}(\sigma, \tau) := \{(\sigma, 1), (\tau, 1)\} \qquad \text{and} \qquad \text{o.p.}(\sigma, \tau) := \text{u.p.}\big(\text{u.p.}(\sigma, \sigma), \text{u.p.}(\sigma, \tau)\big).$$

Fact 4.2. *For any forcing order $\mathbb{P} = (P, \leq, 1)$, for any nonempty filter G on $\mathbb{P}$, and for any P-names σ and τ,*

(i) u.p.(σ, τ) *is a P-name, and* $(\text{u.p.}(\sigma, \tau))_G = \{\sigma_G, \tau_G\}$*;*
(ii) o.p.(σ, τ) *is a P-name, and* $(\text{o.p.}(\sigma, \tau))_G = (\sigma_G, \tau_G)$*.*

Theorem 4.3. *If M is a c.t.m. of* ZFC*, $\mathbb{P} = (P, \leq, 1) \in M$ is a forcing order, and $G \subseteq P$ is a filter that is $\mathbb{P}$-generic over M, then $M[G]$ is a model of* ZFC*.*

Proof. Let M, $\mathbb{P}$, and G satisfy the assumptions of the theorem. To prove that the axioms of $\mathsf{ZF} - \mathsf{Inf}$ hold in $M[G]$, we will use Theorem 2.5 in the lecture notes "Models of Set Theory".

▶ Ext and Fnd hold in $M[G]$, because $M[G]$ is a transitive set (see Theorem 2.8),

▶ Pair holds in $M[G]$: Let $x, y \in M[G]$. It suffices to find $z \in M[G]$ such that $x, y \in z$.

- $x = \sigma_G$ and $y = \tau_G$ for some $\sigma, \tau \in M^P$.
- For $\zeta = \text{u.p.}(\sigma, \tau)$, we have (by Fact 4.2) that $\zeta \in M^P$, so $z := \zeta_G \in M[G]$, and $z = \zeta_G = \{\sigma_G, \tau_G\} = \{x, y\}$.

▶ Uni holds in $M[G]$: Let $x \in M[G]$, that is, $x = \sigma_G$ for some $\sigma \in M^P$. It suffices to find $\tau_G \in M[G]$ with $\tau \in M^P$ such that $\bigcup x \subseteq \tau_G$.

- $\text{dmn}(\sigma) \in M$ is a set of P-names, therefore $\tau := \bigcup \text{dmn}(\sigma) \in M^P$. Hence, $\tau_G \in M[G]$.
- **Claim.** $\bigcup x \subseteq \tau_G$.
  - Let $y \in \bigcup x$. Then $y \in z \in x = \sigma_G$ for some z, where $y, z \in M[G]$ as $M[G]$ is transitive.
  - There exists $(\xi, s) \in \sigma$ with $s \in G$ such that $z = \xi_G$, and there exists $(\rho, r) \in \xi$ with $r \in G$ such that $y = \rho_G$.
  - $(\rho, r) \in \xi \in \text{dmn}(\sigma)$, therefore $(\rho, r) \in \bigcup \text{dmn}(\sigma) = \tau$, where $r \in G$.
  - $y = \rho_G \in \tau_G$

▶ Cmpr holds in $M[G]$: Let $\varphi(z, w_1, \ldots, w_n)$ be a formula with all free variables among $z, w_1, \ldots, w_n$, and let $\sigma_G, \tau_{1G}, \ldots, \tau_{nG}$ be arbitrary elements of $M[G]$ ($\sigma, \tau_1, \ldots, \tau_n \in M^P$). It suffices to show that the following set is a member of $M[G]$:

$$y := \{z \in \sigma_G : \varphi^{M[G]}(z, \tau_{1G}, \ldots, \tau_{nG})\}.$$

- Clearly,
$$\rho := \{(\pi, p) \in \operatorname{dmn}(\sigma) \times P : \big(p \Vdash^* (\pi \in \sigma \wedge \varphi(\pi, \tau_1, \ldots, \tau_n))\big)^M\} \in M^P.$$
- **Claim.** $y = \rho_G$.
  - First, let $x \in \rho_G$; i.e., $x = \pi_G$ for some $(\pi, p) \in \operatorname{dmn}(\sigma) \times P$ with $p \in G$ such that $\big(p \Vdash^* (\pi \in \sigma \wedge \varphi(\pi, \tau_1, \ldots, \tau_n))\big)^M$.
  - $p \Vdash (\pi \in \sigma \wedge \varphi(\pi, \tau_1, \ldots, \tau_n))$, by Corollary 3.18.
  - $\pi_G \in \sigma_G$ and $\varphi^{M[G]}(\pi_G, \tau_{1G}, \ldots, \tau_{nG})$, by $p \in G$ and the definition of $\Vdash$.
  - Thus $x = \pi_G \in y$, proving $y \supseteq \rho_G$.
  - Conversely, let $x \in y$; that is, $x \in \sigma_G$ and $\varphi^{M[G]}(x, \tau_{1G}, \ldots, \tau_{nG})$.
  - $x = \pi_G$ for some $(\pi, q) \in \sigma$ with $q \in G$, so $\pi_G \in \sigma_G \wedge \varphi^{M[G]}(\pi_G, \tau_{1G}, \ldots, \tau_{nG})$; that is, $\pi_G \in \sigma_G \wedge \varphi(\pi_G, \tau_{1G}, \ldots, \tau_{nG})$ holds in $M[G]$.
  - $\big(p \Vdash^* (\pi \in \sigma \wedge \varphi(\pi, \tau_1, \ldots, \tau_n))\big)^M$ for some $p \in G$, by the Forcing Theorem.
  - Thus, $(\pi, p) \in \rho$ with $p \in G$, so $x = \pi_G \in \rho_G$, proving $y \subseteq \rho_G$.

▶ Pset, Repl also hold in $M[G]$: This can be proved with the same techniques as Cmpr. The argument for Repl is more difficult than those for Cmpr and Pset.

▶ Inf holds in $M[G]$: We have proved so far that $M[G]$ is a transitive model of $\mathsf{ZF} - \mathsf{Inf}$. Therefore, by Theorem 3.17 in the lecture notes "Models of Set Theory", it suffices to show that $\omega \in M[G]$. But this holds, because $M \subseteq M[G]$ (by Theorem 2.8) and $\omega \in M$ (by Theorem 3.18 in the lecture notes "Models of Set Theory").

▶ AC holds in $M[G]$: We will prove the equivalent CFP (= Choice Function Principle).[2] Let A be a set of nonempty sets in $M[G]$. We need to show: there is a choice function g for A.

- $\bigcup A \in M[G]$; say $\bigcup A = \sigma_G$ for some $\sigma \in M^P$.
- $\operatorname{dmn}(\sigma) \in M$ and $\operatorname{dmn}(\sigma)$ is a set of P-names in M.
- <u>In M</u>, there exist a cardinal κ and a bijection $f\colon \kappa \to \operatorname{dmn}(\sigma)$.
- Let $\tau := \{\text{o.p.}(\check{\alpha}, f(\alpha)) : \alpha < \kappa\} \times \{1\}$. Clearly, $\tau \in M^P$.
- $\tau_G = \big\{\big(\alpha, (f(\alpha))_G\big)\colon \alpha < \kappa\big\}$ is a function with domain κ in $M[G]$.
- **Claim.** Every $y \in \bigcup A$ is of the form $y = \tau_G(\alpha)$ for some ordinal $\alpha < \kappa$.
  - Let $y \in \bigcup A = \sigma_G$. Then $y = \xi_G$ for some $(\xi, p) \in \sigma$ with $p \in G$.
  - $\xi \in \operatorname{dmn}(\sigma)$, therefore $\xi = f(\alpha)$ for some $\alpha < \kappa$.
  - $y = \xi_G = (f(\alpha))_G = \tau_G(\alpha)$.
- For each $x \in A$ there exists $\alpha < \kappa$ such that $\tau_G(\alpha) \in x$. Let α_x be the least such α. Then the function g defined for all $x \in A$ by $\tau_G(\alpha_x)$ is a choice function for A. $\square$

[2] See Theorem 1.2 in the lecture notes "The Axiom of Choice. Cardinals and Cardinal Arithmetic".

5. Preservation of Cardinals

Let M be a c.t.m. of ZFC, let $\mathbb{P} = (P, \leq, 1) \in M$ be a forcing order, and let $G \subseteq P$ be a filter $\mathbb{P}$-generic over M. We know from Theorems 2.8 and 4.3 that $M[G]$ is also a c.t.m. of ZFC such that $M \subseteq M[G]$ and $G \in M[G]$. We also saw in Theorem 2.11 that M and $M[G]$ have the same ordinals among its members. In particular, ω is the member of both,[3] and so are all finite ordinals (the members of ω). Thus, ω is the least infinite cardinal (=initial ordinal) in both M and $M[G]$.

However, cardinals (= initial ordinals) $> \omega$ might be different in M and $M[G]$. Let α be an ordinal in M. If α is not a cardinal in M, i.e., there exists a bijection $\alpha \to \beta$ in M for some $\beta \in \alpha$, then by absoluteness[4], the same holds in $M[G]$. However, it may happen that such a bijection does not exist in M, but it does exist in $M[G]$. In particular, it can happen that the least uncountable cardinal ω_1^M in M is not the same ordinal as the least uncountable cardinal $\omega_1^{M[G]}$ in $M[G]$.[5]

Definition 5.1. Let M be a c.t.m. of ZFC, let κ be an infinite cardinal in M, and let $\mathbb{P} \in M$ be a forcing order. We say that

- $\mathbb{P}$ *preserves cardinals* $\geq \kappa$ if for every filter $G \subseteq P$ that is $\mathbb{P}$-generic over M, and for every ordinal $\alpha \geq \kappa$ in M, if α is a cardinal in M then α is a cardinal in $M[G]$;
- $\mathbb{P}$ *preserves cofinalities* $\geq \kappa$ if for every filter $G \subseteq P$ that is $\mathbb{P}$-generic over M, and for every limit ordinal α in M such that $\mathrm{cf}^M(\alpha) \geq \kappa$, we have $\mathrm{cf}^M(\alpha) = \mathrm{cf}^{M[G]}(\alpha)$;
- $\mathbb{P}$ *preserves regular cardinals* $\geq \kappa$ if for every filter $G \subseteq P$ that is $\mathbb{P}$-generic over M, and for every ordinal $\alpha \geq \kappa$ in M, if α is a regular cardinal in M then α is a regular cardinal in $M[G]$.

We will say that $\mathbb{P}$ *preserves cardinals* [*cofinalities*, *regular cardinals*] to mean that $\mathbb{P}$ preserves cardinals [cofinalities, regular cardinals] $\geq \omega$.

---

[3] See Theorem 3.18 in the lecture notes "Models of Set Theory".

[4] See Corollary 3.16 in the lecture notes "Models of Set Theory".

[5] For a specific example, see p. 207 of "Lectures on Set Theory" by Donald J. Monk.

Lemma 5.2. *Let M be a c.t.m. of* ZFC*, let κ be an infinite cardinal in M, and let $\mathbb{P} \in M$ be a forcing order.*

(i) *If $\mathbb{P}$ preserves regular cardinals $\geq \kappa$, then $\mathbb{P}$ preserves cofinalities $\geq \kappa$.*
(ii) *If $\mathbb{P}$ preserves cofinalities $\geq \kappa$, and κ is regular, then $\mathbb{P}$ preserves cardinals $\geq \kappa$.*

Proof. Let $G \subseteq P$ be any filter that is $\mathbb{P}$-generic over M.

(i) Let $\alpha \geq \kappa$ be a limit ordinal in M with $\mathrm{cf}^M(\alpha) \geq \kappa$.

- $\mathrm{cf}^M(\alpha)$ is a regular cardinal in M, and hence (by our assumption) in $M[G]$.
- In M, and hence in $M[G]$, there exists a strictly increasing function $f\colon \mathrm{cf}^M(\alpha) \to \alpha$ such that $\mathrm{rng}(f)$ is unbounded in α.
- In $M[G]$, there exists a strictly increasing function $g\colon \mathrm{cf}^{M[G]}(\alpha) \to \alpha$ such that $\mathrm{rng}(g)$ is unbounded in α.
- $\mathrm{cf}^M(\alpha) = \mathrm{cf}^{M[G]}(\alpha)$, by Corollary 4.13 in "The Axiom of Choice. Cardinals ...".

(ii) Suppose the assumptions hold, but there is a cardinal $\lambda \geq \kappa$ in M which is not a cardinal in $M[G]$. Choose λ smallest with these properties.

- Case 1: λ is regular in M. Then
  - $\lambda = \mathrm{cf}^M(\lambda) \overset{!}{=} \mathrm{cf}^{M[G]}(\lambda)$, where $\overset{!}{=}$ holds by our assumption;
  - λ is a regular cardinal in $M[G]$, contradicting the choice of λ.
- Case 2: λ is singular in M. Then
  - $\lambda > \kappa$ (as κ is regular);
  - for every μ with $\kappa \leq \mu < \lambda$, μ is a cardinal in M iff μ is a cardinal in $M[G]$;
  - $\lambda = \bigcup\{\mu : \kappa \leq \mu < \lambda, \mu \text{ is a cardinal}\}$ in M, hence in $M[G]$, so λ is a cardinal in $M[G]$; this contradicts the choice of λ. □

We will be able to establish cardinality/cofinality preservation properties for forcing orders $\mathbb{P}$ satisfying an additional property, which we introduce now.

Definition 5.3. Let $\mathbb{P}$ be a forcing order, and let κ be a cardinal. We say that $\mathbb{P}$ satisfies the *κ-chain condition* (abbreviated *κ-c.c.*), if every antichain[6] in $\mathbb{P}$ has cardinality $< \kappa$.

The ω_1-chain condition is called *countable chain condition* (abbreviated *c.c.c.*).

[6]Recall from Definition 1.2 that an antichain in $\mathbb{P} = (P, \leq, 1)$ is a set $A \subseteq P$ such that any two distinct elements of A are incompatible.

To show that a cardinal λ in a c.t.m. M of ZFC remains a cardinal in $M[G]$, that is, no bijection $\lambda \to \alpha$ with $\alpha \in \lambda$ is created in the passage from M to $M[G]$, we have to study the relationship between functions $A \to B$ in M and functions $A \to B$ in $M[G]$ where $A, B \in M$.

Theorem 5.4. *Let M be a c.t.m. of ZFC, let κ be an infinite cardinal in M, let $\mathbb{P} \in M$ be a forcing order such that $\mathbb{P}$ satisfies the κ-c.c. in M, and let $G \subseteq P$ be a filter that is $\mathbb{P}$-generic over M. If $A, B \in M$, and $f \in M[G]$ is a function $A \to B$, then there exists $F \in M$ such that F is function $A \to \mathcal{P}(B)$ with*

(i) *$f(a) \in F(a)$ for all $a \in A$, and*
(ii) *$(|F(a)| < \kappa)^M$ for all $a \in A$.*[7]

Proof. Let $f\colon A \to B$ be a function in $M[G]$ with $A, B \in M$. Then $f = \tau_G$ for some $\tau \in M^P$, and the statement "τ_G is a function $\check{A}_G \to \check{B}_G$" holds in $M[G]$. Hence, by Corollary 3.19,

$$(\dagger) \qquad p \Vdash \text{“}\tau \text{ is a function } \check{A} \to \check{B}\text{”} \qquad \text{for some } p \in G.$$

Now we define a function $F\colon A \to \mathcal{P}(B)$ for all $a \in A$ by

$$\begin{aligned} F(a) &:= \{b \in B : \text{there exists } q \leq p \text{ such that } (q \Vdash^* \text{o.p.}(\check{a}, \check{b}) \in \tau)^M\} \\ &\stackrel{*}{=} \{b \in B : \text{there exists } q \leq p \text{ such that } q \Vdash \text{o.p.}(\check{a}, \check{b}) \in \tau\}, \end{aligned}$$

where $\stackrel{*}{=}$ holds by Corollary 3.18. We have $F \in M$ by the first description of $F(a)$ and Repl.

(i) Let $a \in A$ and $b := f(a)$.

- $(\text{o.p.}(\check{a}, \check{b}))_G = (\check{a}_G, \check{b}_G) = (a, b) \in f = \tau_G$, so $r \Vdash \text{o.p.}(\check{a}, \check{b}) \in \tau$ for some $r \in G$, by Corollary 3.19.
- $q \leq p, r$ for some $q \in G$, so $q \Vdash \text{o.p.}(\check{a}, \check{b}) \in \tau$, proving $f(a) = b \in F(a)$.

(ii) Let $a \in A$. We will prove $(|F(a)| < \kappa)^M$ by finding a one-to-one function $Q\colon F(a) \to P$ in M whose range is an antichain in $\mathbb{P}$. By the κ-c.c. in $\mathbb{P}$, this will complete the proof.

- By AC in M, there exists a function $Q\colon F(a) \to P$ such that for every $b \in F(a)$ we have $Q(b) \leq p$ and $Q(b) \Vdash \text{o.p.}(\check{a}, \check{b}) \in \tau$.
- **Claim.** If $b, c \in F(a)$ and $b \neq c$, then $Q(b) \perp Q(c)$.
  Let $b, c \in F(a)$ satisfy $Q(b) \not\perp Q(c)$; we want to conclude that $b = c$.
  - There exists $r \in P$ with $r \leq Q(b), Q(c)$, so $r \Vdash \text{o.p.}(\check{a}, \check{b}) \in \tau \wedge \text{o.p.}(\check{a}, \check{c}) \in \tau$.
  - $r \Vdash \big(\text{o.p.}(\check{a}, \check{b}) \in \tau \wedge \text{o.p.}(\check{a}, \check{c}) \in \tau\big) \to \check{b} = \check{c}$, by $(\dagger)$.
  - Hence, $r \Vdash \check{b} = \check{c}$, by the definition of $\Vdash$.
  - There is a filter $H \subseteq P$ with $r \in H$ which is $\mathbb{P}$-generic over M (Theorem 1.5).
  - $b = \check{b}_H = \check{c}_H = c$. ∎

[7] Later on, when we work in M and μ, κ are cardinals in M, we may write $\mu <^M \kappa$ instead of $(\mu < \kappa)^M$.

Theorem 5.5. *Let M be a c.t.m. of* ZFC*, let κ be an infinite cardinal in M, and let $\mathbb{P} \in M$ be a forcing order which satisfies κ-c.c. in M. Then:*

(i) *$\mathbb{P}$ preserves regular cardinals $\geq \kappa$ and cofinalities $\geq \kappa$.*
(ii) *If κ is a regular cardinal in M, then $\mathbb{P}$ preserves cardinals $\geq \kappa$.*

Proof. By Lemma 5.2, it suffices to show that $\mathbb{P}$ preserves regular cardinals $\geq \kappa$. Assume that there is a regular cardinal $\lambda \geq \kappa$ in M which is not a regular cardinal in $M[G]$.

- In $M[G]$, λ is a limit ordinal which is not a regular cardinal, therefore there exist $\alpha \in \lambda$ and a strictly increasing function $f\colon \alpha \to \lambda$ such that $\operatorname{rng}(f)$ is unbounded in λ.
- In M, there exists $F\colon \alpha \to \mathcal{P}(\lambda)$ such that $f(\xi) \in F(\xi)$ and $|F(\xi)| <^M \kappa$ for all $\xi < \alpha$ (by Theorem 5.4).
- $S := \bigcup_{\xi<\alpha} F(\xi) \in M$, $S \subseteq \lambda$, S is unbounded in λ, and $|S| \leq^M \sum_{\xi<\alpha} |F(\xi)| <^M \lambda = \operatorname{cf}^M(\lambda)$ (as λ is a regular cardinal in M); contradiction. $\square$

6. The Consistency of ZFC + ¬CH

Notation 6.1. If I and J are sets and λ is an infinite cardinal, let

$$\mathrm{Fn}(I,J,\lambda) := \{f \subseteq I \times J : f \text{ is a function and } |f| < \lambda\}, \quad \text{and}$$
$$\mathbb{F}\mathrm{n}(I,J,\lambda) := \big(\mathrm{Fn}(I,J,\lambda), \supseteq, \emptyset\big).$$

Clearly, $\mathbb{F}\mathrm{n}(I,J,\lambda)$ is a forcing order.

Theorem 6.2. (Cohen) *Let M be a c.t.m. of* ZFC*, let κ be an infinite cardinal in M, and let $G \subseteq \mathrm{Fn}^M(\kappa \times \omega, 2, \omega)$ be a filter that is $\mathbb{F}\mathrm{n}^M(\kappa \times \omega, 2, \omega)$-generic over M. Then*

(i) *$M[G]$ is a c.t.m. of* ZFC *with the same ordinals as M;*
(ii) *$M[G]$ has the same cardinals and the same cofinalities of limit ordinals as M; and*
(iii) *$(2^\omega \geq \kappa)^{M[G]}$ (i.e., $2^\omega \geq \kappa$ holds in $M[G]$).*

Proof. (i) This was established in Theorems 4.3 and 2.11.

(ii) By Theorem 5.5, it suffices to prove that $\mathbb{F}\mathrm{n}^M(\kappa \times \omega, 2, \omega)$ has c.c.c. (=ω_1-c.c.). This will follow if we prove the following claim (in ZFC):

Claim 6.3. *If K is an infinite set, then $\mathbb{F}\mathrm{n}(K, 2, \omega)$ satisfies c.c.c.*

Proof of Claim 6.3. Let $\mathcal{F} \subseteq \mathrm{Fn}(K,2,\omega)$ be uncountable. Our goal is to show that $\mathcal{F}$ is not an antichain, i.e., there exist distinct $f, g \in \mathcal{F}$ which are compatible.

- $\{\mathrm{dmn}(f) : f \in \mathcal{F}\}$ is an uncountable family of finite subsets of K, because
  - $\mathrm{dmn}(f) < \omega$ for all $f \in \mathcal{F}$, and
  - for each finite set $D \subseteq K$, there are only finitely many $f \in \mathcal{F}$ with $\mathrm{dmn}(f) = D$.
- By the Δ-System Theorem,[8] there exists a Δ-system $\mathcal{D} \subseteq \{\mathrm{dmn}(f) : f \in \mathcal{F}\}$ such that $\mathcal{D}$ is also uncountable. Let R be the root of $\mathcal{D}$; i.e., $A \cap B = R$ for any two distinct $A, B \in \mathcal{D}$.
- Let $\mathcal{G} = \{f \in \mathcal{F} : \mathrm{dmn}(f) \in \mathcal{D}\}$. Then
  - $\mathcal{G}$ is uncountable, because $\mathcal{G} \to \mathcal{D}$, $f \mapsto \mathrm{dmn}(f)$ is onto; moreover,
  $$\mathcal{G} = \bigcup_{h \in {}^R 2} \{f \in \mathcal{G} : f \restriction R = h\}.$$
- There exists $h \in {}^R 2$ such that $\mathcal{G}_h := \{f \in \mathcal{G} : f \restriction R = h\}$ is uncountable, since $|\mathcal{G}| > \omega$ and $|{}^R 2| < \omega$.
- Any two elements of $\mathcal{G}_h$ are compatible. $\diamond$

[8] See Corollary 1.3 in the lecture notes "Infinite Combinatorics".

(iii) We have to show that $M[G]$ contains a set which is a one-to-one function $\kappa \to {}^{\omega}2$ in $M[G]$. Let $g := \bigcup G$. Our goal is to show that g is a function $\kappa \times \omega \to 2$ in $M[G]$, which induces an injection $\kappa \to {}^{\omega}2$ in $M[G]$.

- $g \in M[G]$, and g is a function $\subseteq (\kappa \times \omega) \times 2$, because any two elements of G are compatible.
- For each pair $(\alpha, m) \in \kappa \times \omega$ let
  $$D_{\alpha,m} := \{f \in \mathrm{Fn}^M(\kappa \times \omega, 2, \omega) : (\alpha, m) \in \mathrm{dmn}(f)\}.$$
  Then
  - $D_{\alpha,m} \in M$, and
  - $D_{\alpha,m}$ is dense in $\mathbb{F}\mathrm{n}^M(\kappa \times \omega, 2, \omega)$, because for any $f_0 \in \mathrm{Fn}^M(\kappa \times \omega, 2, \omega)$,
    - ▷ either $f := f_0 \in D_{\alpha,m}$ and clearly $f \supseteq f_0$,
    - ▷ or $f := f_0 \cup \{((\alpha, m), 0)\} \in D_{\alpha,m}$ with $f \supseteq f_0$.
- It follows that $\mathrm{dmn}(g) = \kappa \times \omega$, because $G \cap D_{\alpha,m} \neq \emptyset$ for every $(\alpha, m) \in \kappa \times \omega$ (since G is $\mathbb{F}\mathrm{n}^M(\kappa \times \omega, 2, \omega)$-generic over M).
- Now, for any distinct $\alpha, \beta \in \kappa$, let
  $$E_{\alpha,\beta} := \{f \in \mathrm{Fn}^M(\kappa \times \omega, 2, \omega) : \text{there exists } m \in \omega \text{ with } f(\alpha, m) \neq f(\beta, m)\}.$$
  Then
  - $E_{\alpha,\beta} \in M$, and
  - $E_{\alpha,\beta}$ is dense in $\mathbb{F}\mathrm{n}^M(\kappa \times \omega, 2, \omega)$, because for any $f_0 \in \mathrm{Fn}^M(\kappa \times \omega, 2, \omega)$
    - ▷ there exists $m \in \omega$ such that $(\alpha, m), (\beta, m) \notin \mathrm{dmn}(f_0)$ (as $\mathrm{dmn}(f_0)$ is finite), so
      $$f := f_0 \cup \{((\alpha, m), 0), ((\beta, m), 1)\} \in E_{\alpha,\beta}$$
      with $f \supseteq f_0$.
- For any distinct $\alpha, \beta \in \kappa$, the functions $g(\alpha, -), g(\beta, -)$ are different, because $G \cap E_{\alpha,\beta} \neq \emptyset$, so for any $f \in G \cap E_{\alpha,\beta}$ we have $g \supseteq f$ and $f(\alpha, m) \neq f(\beta, m)$ for some $m \in \omega$.
- Thus, the function $\kappa \to {}^{\omega}2$, $\alpha \mapsto g(\alpha, -)$ is one-to-one. □

Corollary 6.4. (Cohen) *If* ZFC *is consistent, then so is* ZFC $+ \neg$CH*.*

Proof. Assume that ZFC is consistent, and let M be a c.t.m. of ZFC. Apply Theorem 6.2 with $\kappa = \omega_2^M$.

- $M[G]$ is a model of ZFC, by part (i) of the theorem.
- $\omega^{M[G]} = \omega^M$, by absoluteness.
- $\omega_1^{M[G]} = \omega_1^M$ and $\omega_2^{M[G]} = \omega_2^M$, since M and $M[G]$ have the same cardinals by part (ii) of the theorem.
- $M[G]$ is a model of $\neg$CH, because $(2^\omega \geq \omega_2^M)^{M[G]}$ by part (iii) of the theorem, and $(2^\omega \geq \omega_2^M)^{M[G]}$ is the same statement as $(2^\omega \geq \omega_2)^{M[G]}$. □

7. The Consistency of ZFC + CH (Brief sketch)

Theorem 7.1. (Gödel) *If* ZFC *is consistent, then so is* ZFC + CH.

This theorem, together with Cohen's result (Corollary 6.4) proves the independence of CH from ZFC.

Theorem 7.1 is a consequence of a much stronger result of Gödel (1940), which proves that if ZF is consistent, then so is ZFC + GCH. Gödel introduced the notion of 'constructible sets'[9] to prove his theorem.

Here we sketch how forcing can be used to prove Theorem 7.1. The argument follows the same main steps that led to the proof of the consistency of $\neg$CH.

Step 1. Let M be a c.t.m. of ZFC, let κ be an infinite cardinal in M, and let $\mathbb{P} \in M$ be a forcing order. The terminology $\mathbb{P}$ *preserves cardinals* $\geq \kappa$, $\mathbb{P}$ *preserves cofinalities* $\geq \kappa$, $\mathbb{P}$ *preserves regular cardinals* $\geq \kappa$ introduced in Definition 5.1 can also be used for $\leq \kappa$ instead of $\geq \kappa$, and the analogues of the statements in Lemma 5.2 follow similarly for $\leq \kappa$ as well.

Step 2. Next we define a condition on forcing orders which ensures the cardinal preservation property required for the argument.

Definition 7.2. Let $\mathbb{P} = (P, \leq, 1)$ be a forcing order, and let λ be an infinite cardinal. We say that $\mathbb{P}$ is λ-*closed* if for all $\gamma < \lambda$ and for any system $\langle p_\xi : \xi < \gamma \rangle$ of elements of P such that $p_\eta \leq p_\xi$ whenever $\xi < \eta < \gamma$, there exists $q \in P$ such that $q \leq p_\xi$ for all $\xi < \gamma$.

Step 3. We have the following analogs of Theorems 5.4 and Theorem 5.5.

Theorem 7.3. *Let M be a c.t.m. of* ZFC*, let λ be an infinite cardinal in M, let $\mathbb{P} \in M$ be a forcing order such that $\mathbb{P}$ is λ-closed in M, and let $G \subseteq P$ be a filter that is $\mathbb{P}$-generic over M. If $A, B \in M$ where $(|A| < \lambda)^M$, and $f \in M[G]$ is a function $A \to B$, then $f \in M$.*

Theorem 7.4. *Let M be a c.t.m. of* ZFC*, let λ be an infinite cardinal in M, and let $\mathbb{P} \in M$ be a forcing order such that $\mathbb{P}$ is λ-closed in M. Then $\mathbb{P}$ preserves cardinals $\leq \lambda$ and cofinalities $\leq \lambda$.*

Step 4. Finally, the analog of Theorem 6.2 is:

Theorem 7.5. *Let M be a c.t.m. of* ZFC*, and let $G \subseteq \mathrm{Fn}^M(\omega_1^M \times \omega, 2, \omega_1^M)$ be a filter that is $\mathbb{F}\mathrm{n}^M(\omega_1^M \times \omega, 2, \omega_1^M)$-generic over M. Then*

(i) *$M[G]$ is a c.t.m. of* ZFC *with the same ordinals as M;*
(ii) *$\omega_1^{M[G]} = \omega_1^M$; and*
(iii) CH *holds in $M[G]$.*

This completes the proof of Theorem 7.1.

[9]See Section 23 in "Lectures in Set Theory" by J. Donald Monk.