Definition 3.1. Let M be a c.t.m. of ZFC, and let $\mathbb{P} = (P, \leq, 1) \in M$ be a forcing order. For each formula $\varphi(v_0, \ldots, v_{m-1})$ in the language of set theory, we define another formula

$$p \Vdash_{\mathbb{P},M} \varphi(\sigma_0, \ldots, \sigma_{m-1})$$

[read: *p forces* $\varphi(\sigma_0, \ldots, \sigma_{m-1})$ *with respect to* $\mathbb{P}$ *and* M],

which states that

> $\mathbb{P}$ is a forcing order, $\mathbb{P} \in M$, $\sigma_0, \ldots, \sigma_{m-1} \in M^P$, $p \in P$, and for every filter $G \subseteq P$ which is $\mathbb{P}$-generic over M, if $p \in G$, then the formula $\varphi^{M[G]}(v_0, \ldots, v_{m-1})$ (= the relativization of φ to $M[G]$) holds for the elements $\sigma_{0G}, \ldots, \sigma_{(m-1)G}$.

Theorem 3.8. *Let* $\mathbb{P} = (P, \leq, 1)$ *be a forcing order, and let* $p, q \in P$ *and* $a, b \in \mathrm{RO}(\mathbb{P})$.

- (i) $e[P]$ *is dense in* $\mathrm{RO}(\mathbb{P}) \setminus \{\emptyset\}$, *i.e., for any nonempty set* $Y \in \mathrm{RO}(\mathbb{P})$ *there exists* $p \in P$ *such that* $e(p) \subseteq Y$.
- (ii) $e(p) = \mathrm{int}\big(\mathrm{cl}(P \downarrow p)\big) = \{r \in P : \textit{for all } u \in P \textit{ with } u \leq r,\ u \textit{ and } p \textit{ are compatible}\}$. *Hence,* $p \perp q$ *iff* $e(p) \cap e(q) = \emptyset$.
- (iii) *The following conditions on* p, q *are equivalent:*
  - (a) $e(p) \subseteq e(q)$;
  - (b) $\{r \in P : r \leq p, q\}$ *is dense below* p.

  Hence $p \leq q$ *implies that* $e(p) \subseteq e(q)$, *and* $e(p) \subseteq e(q)$ *implies that* p, q *are compatible.*

Definition 3.12. Let $\mathbb{P}$ be a forcing order, and let $p \in P$. For arbitrary formula φ (with all free variables among $v_0, \ldots, v_{m-1}$) and for all $\sigma_0, \ldots, \sigma_{m-1} \in \mathbf{V}^P$, we define $p \Vdash^* \varphi(\sigma_0, \ldots, \sigma_{m-1})$ to mean that $e(p) \subseteq [\![\varphi(\sigma_0, \ldots, \sigma_{m-1})]\!]$.

The Forcing Theorem 3.13. *Let* M *be a c.t.m of* ZFC, *let* $\mathbb{P} \in M$ *be a forcing order, and let* $G \subseteq P$ *be a filter that is* $\mathbb{P}$*-generic over* M. *For any formula* φ *(with all free variables among* $v_0, \ldots, v_{m-1}$*) and for any* $\sigma_0, \ldots, \sigma_{m-1} \in M^P$, *the following conditions are equivalent:*

- (a) $\varphi(\sigma_{0G}, \ldots, \sigma_{(m-1)G})$ *holds in* $M[G]$.
- (b) *There is a* $p \in G$ *such that* $\big(p \Vdash^* \varphi(\sigma_0, \ldots, \sigma_{m-1})\big)^M$.

Proof of the Forcing Theorem. Most work goes into proving the equivalence of (a)–(b) for atomic formulas. This is done by induction on the class relation $\mathbf{R}$ used in the simultaneous definitions of $[\![\sigma = \tau]\!]$ and $[\![\sigma \in \tau]\!]$ (by recursion on $\mathbf{R}$) in the proof of Theorem 3.9.

(b) $\Rightarrow$ (a) for $v_0 = v_1$

Let $\sigma, \tau \in M^P$, and assume that there exists $p \in G$ such that $(p \Vdash^* \sigma = \tau)^M$, that is, such that <u>in M</u> we have

$$e(p) \subseteq [\![\sigma = \tau]\!] = \bigwedge_{(\xi,r)\in\tau} (e(r)' \vee [\![\xi \in \sigma]\!]) \wedge \bigwedge_{(\eta,q)\in\sigma} (e(q)' \vee [\![\eta \in \tau]\!]). \tag{1}$$

Our goal is to show that $\sigma_G = \tau_G$. By symmetry, it suffices to argue that $\sigma_G \subseteq \tau_G$.
Let $a \in \sigma_G = \{\eta_G : (\eta, q) \in \sigma \text{ for some } q \in G\}$; say $a = \eta_G$ with $(\eta, q) \in \sigma$, $q \in G$. Then:

① $e(p) \subseteq e(q)' \vee [\![\eta \in \tau]\!]$, by (1).
② $e(p) \wedge e(q) \subseteq [\![\eta \in \tau]\!]$, by ① ($\wedge$ both sides with $e(q)$, and use the distributive law).
③ there exists $r \in G$ with $r \leq p, q$, since G is a filter on $\mathbb{P}$.
④ $e(r) \subseteq e(p) \wedge e(q) \subseteq [\![\eta \in \tau]\!]$, by ③ combined with Theorem 3.8(iii), and by ②.
⑤ $(r \Vdash^* \eta \in \tau)^M$, by the definition of $(\Vdash^*)^M$.
⑥ $a = \eta_G \in \tau_G$, by the induction hypothesis.