Set Theory (MATH 6730)

Ordinals. Transfinite Induction and Recursion

We will not use the Axiom of Choice in this chapter. All theorems we prove will be theorems of $\mathsf{ZF} = \mathsf{ZFC} \setminus \{\mathsf{AC}\}$.

1. Ordinals

To motivate ordinals, recall from elementary (naive) set theory that natural numbers can be used to label elements of a finite set to indicate a linear ordering:

$$0\text{th},\ \ 1\text{st},\ \ 2\text{nd},\ \ \ldots,\ \ 2019\text{th element}.$$

Although this will not be immediately clear from the definition, 'ordinal numbers' or 'ordinals' are sets which are introduced to play the same role for special linear orderings, called 'well-orderings', of arbitrary sets. For example, one such ordering might be the following:

$$\underbrace{0\text{th},\ \ 1\text{st},\ \ 2\text{nd},\ \ \ldots,\ \ 2019\text{th},\ \ \ldots}_{\text{we have run out of all natural numbers}},\ \ \omega\text{th},\ \ (\omega+1)\text{st element}.$$

Definition 1.1. A set s is called *transitive*, if every element of s is a subset of s; i.e., for any sets x, y such that $x \in y \in s$ we have that $x \in s$. Thus, the class of all transitive sets is defined by the formula

$$\mathrm{tr}(s) \equiv \forall x\, \forall y\, \big((x \in y \ \wedge\ y \in s) \to x \in s\big).$$

Definition 1.2. A transitive set of transitive sets is called an *ordinal number* or *ordinal*. Thus, the class **On** of all ordinals is defined by the formula

$$\mathrm{on}(x) \equiv \mathrm{tr}(x) \wedge \forall y \in x\, \mathrm{tr}(y).$$

Ordinals will usually be denoted by the first few letters of the Greek alphabet.

The following facts are easy consequences of the definition of ordinals.

Theorem 1.3.

(i) $\emptyset \in \mathbf{On}$.

(ii) *If* $\alpha \in \mathbf{On}$, *then* $\alpha \cup \{\alpha\} \in \mathbf{On}$.

(iii) *If* $\alpha \in \mathbf{On}$ *then* $\alpha \subseteq \mathbf{On}$; *i.e., every member of an ordinal is an ordinal.*

(iv) *If* A *is a set of ordinals, then* $\bigcup A \in \mathbf{On}$.

(v) *If* A *is a nonempty set of ordinals, then* $\bigcap A \in \mathbf{On}$.

Theorem 1.3(iii) shows that if **On** was a set, it would be an ordinal, and hence we would get $\mathbf{On} \in \mathbf{On}$, which is impossible. This proves:

Corollary 1.4. **On** *is not a set.*

Definition 1.5. Let $\mathbf{A}$ be an ordinal or **On**, and define the class relations $<_{\mathbf{A}}$ and $\leq_{\mathbf{A}}$ on $\mathbf{A}$ by the formulas

$$x <_{\mathbf{A}} y \equiv x \in \mathbf{A} \wedge y \in \mathbf{A} \wedge x \in y \qquad \text{and}$$
$$x \leq_{\mathbf{A}} y \equiv x \in \mathbf{A} \wedge y \in \mathbf{A} \wedge (x \in y \vee x = y).$$

The subscript $_{\mathbf{A}}$ will be omitted if $\mathbf{A}$ is clear from the context.

Theorem 1.6.

(i) *If* $\mathbf{A}$ *is an ordinal or* $\mathbf{A} = \mathbf{On}$*, then* $<$ *is a* (strict) linear order *on* $\mathbf{A}$*; that is,*

$$\mathsf{ZF} \ \vdash \ \underbrace{\forall x \in \mathbf{A}\, \neg\, x < x}_{<\text{ is irreflexive}} \ \wedge \ \underbrace{\forall x \in \mathbf{A}\, \forall y \in \mathbf{A}\, \forall z \in \mathbf{A}\, \big((x < y \wedge y < z) \to x < z\big)}_{<\text{ is transitive}}$$
$$\wedge \ \underbrace{\forall x \in \mathbf{A}\, \forall y \in \mathbf{A}\, (x = y \vee x < y \vee y < x)}_{\text{the trichotomy law holds for }<}.$$

(ii) *For arbitrary ordinals* α *and* β *we have that*
- $\alpha \leq \beta$ *if and only if* $\alpha \subseteq \beta$*, and*
- $\alpha < \beta$ *if and only if* $\alpha \subsetneq \beta$*.*

(iii) *For any set* A *of ordinals,* $\bigcup A$ *is the* least upper bound *for* A *with respect to* $\leq$*; that is, the ordinal* $\bigcup A$ *satisfies*
- $\alpha \leq \bigcup A$ *for all* $\alpha \in A$*, and*
- *for every ordinal* γ *such that* $\alpha \leq \gamma$ *for all* $\alpha \in A$*, we have that* $\bigcup A \leq \gamma$*.*

(iv) *For any nonempty set* A *of ordinals,* $\bigcap A$ *is the* least element *of* A *with respect to* $\leq$*; that is, the ordinal* $\bigcap A$ *satisfies*
- $\bigcap A \in A$*, and*
- $\bigcap A \leq \alpha$ *for all* $\alpha \in A$*.*

(v) *There do not exist ordinals* α, β *such that* $\alpha < \beta < \alpha \cup \{\alpha\}$*.*

(vi) *For arbitrary ordinals* α *and* β *we have that*
- $\alpha < \beta$ *if and only if* $\alpha \cup \{\alpha\} \leq \beta$*.*

Idea of Proof. All statements in (i)–(vi), except the trichotomy of $<$ in (i), are fairly straightforward consequences of the definition of ordinals and properties of ordinals proved in Theorem 1.3 or earlier items of this theorem.

We prove the trichotomy law for $<$ on $\mathbf{A}$ by contradiction. So, assume that the trichotomy law fails for $<$ on $\mathbf{A}$, and choose $\alpha, \beta \in \mathbf{A}$ such that $\tau(\alpha, \beta)$, where $\tau(x, y)$ is the formula

$$\tau(x, y) \equiv \neg\, x = y \;\wedge\; \neg\, x \in y \;\wedge\; \neg\, y \in x.$$

Prove the following claims:

1. The set $A = (\alpha \cup \{\alpha\}) \cup (\beta \cup \{\beta\})$ is an ordinal.
2. The set $C = \{x \in A : \exists y \in A\ \tau(x, y)\}$ has an element γ such that $\gamma \cap C = \emptyset$.

Fix such a γ.

3. The set $D = \{y \in A : \tau(\gamma, y)\}$ has an element δ such that $\delta \cap D = \emptyset$.

Fix such a δ, and note that we have $\tau(\gamma, \delta)$ by construction, so

$$(*) \qquad \gamma \neq \delta, \quad \gamma \notin \delta, \quad \text{and} \quad \delta \notin \gamma.$$

4. Now prove $\gamma = \delta$ by arguing that every element of γ is an element of δ, and every element of δ is an element of γ.

We have reached the desired contradiction, which proves the trichotomy of $<$ on $\mathbf{A}$. $\square$

The statement in Theorem 1.6(v) motivates the following terminology: for every ordinal α, we call the ordinal $\alpha \cup \{\alpha\}$ the *successor* of α, and denote it by $\alpha +' 1$.[1]

By combining statements (i) and (iv) of Theorem 1.6 we get that for every ordinal α, the relation $<$ is a linear order on α such that every nonempty subset of α has a least element. Such a linear order is called a *well-order*.

Since $\emptyset \subseteq x$ for every set, we see from Theorem 1.6(ii) that $\emptyset$ is the least ordinal; therefore we will also denote it by 0 (zero). The successor of 0 is the set $\emptyset \cup \{\emptyset\} = \{\emptyset\}$, which will be denoted by 1, and the successor of 1, which is the set $\{\emptyset\} \cup \{\{\emptyset\}\} = \{\emptyset, \{\emptyset\}\}$, will be denoted by 2, etc.

Definition 1.7. Let $\alpha \in \mathbf{On}$. We will call α a *successor ordinal* if $\alpha = \beta +' 1 = \beta \cup \{\beta\}$ for some ordinal β, and we will call α a *limit ordinal* if $\alpha \neq 0$ and α is not a successor ordinal.

Theorem 1.8. *Let $\alpha \in \mathbf{On}$.*

(i) *The following conditions on α are equivalent:*
   (a) *α is a limit ordinal;*
   (b) *$\alpha \neq 0$ and for every ordinal $\beta < \alpha$ there exists an ordinal γ such that $\beta < \gamma < \alpha$;*
   (c) *$0 \neq \alpha = \bigcup \alpha$.*

(ii) *If α is a successor ordinal and $\alpha = \beta +' 1$, then $\bigcup \alpha = \beta$.*

[1]After introducing addition for ordinals, the ordinal $\alpha +' 1 = \alpha \cup \{\alpha\}$ will turn out to be the sum of α and 1 where 1 is the successor of $0 \stackrel{\text{def}}{=} \emptyset$.

By construction, 1, 2, etc. are successor ordinals. Now we will use the Axiom of Infinity to show that there exists a limit ordinal. Let us call a set u *inductive* if it is a member of the class defined by the (abbreviated) formula

$$\iota(u) \equiv \emptyset \in u \ \wedge\ \forall x \in u\, x \cup \{x\} \in u.$$

The Axiom of Infinity is the statement that there exists an inductive set u. Let

$$\omega = \bigcap \{v \in \mathcal{P}(u) : v \text{ is inductive}\} = \{x \in u : \forall v\, (\iota(v) \to x \in v)\}.$$

To justify the second equality notice that $\supseteq$ is clear, while $\subseteq$ holds, because if w is an inductive set, so is $w \cap u \in \mathcal{P}(u)$. Thus,

- ω is a set (by Comprehension);
- ω is inductive (as the intersection of any nonempty set of inductive sets is inductive); and
- ω is a subset of all inductive sets.

This shows that ω is the least inductive set with respect to $\subseteq$; hence it is independent of the choice of u.

Definition 1.9. The elements of ω are called *natural numbers*, so ω is *the set of all natural numbers.*

The fact that ω has no proper subset that is inductive, is the Induction Principle for ω:

Theorem 1.10. *For every subset A of ω, if*

- *$0 \in A$, and*
- *whenever $y \in A$ then $y +' 1 = y \cup \{y\} \in A$,*

then $A = \omega$.

We can use induction on ω to prove the following properties of ω:

Theorem 1.11. *ω is a limit ordinal. In fact, ω is the smallest limit ordinal with respect to $<$; that is, every ordinal $\beta < \omega$ is either 0 or a successor ordinal.*

2. Well-founded and Set-like Class Relations

Our goal is to extend

- the Induction Principle for ω, and
- the familiar idea of constructing infinite sequences (i.e., functions with domain ω) by recursion

to all ordinals. It turns out that induction and recursion work — and are useful — in a much broader context, which we will discuss now.

Let $\mathbf{R}$ be a class relation, and let $\mathbf{A}$ be a class; let $\mathbf{R}$ be defined by the formula $\varphi(x,y)$, and let $\mathbf{A}$ be defined by the formula $\psi(x)$.

Definition 2.1. We say that $\mathbf{R}$ *is well-founded on* $\mathbf{A}$ if $\mathbf{R} \subseteq \mathbf{A} \times \mathbf{A}$ and for every nonempty sub<u>set</u> X of $\mathbf{A}$ there exists an element $x \in X$ such that there is no $y \in X$ satisfying $(y,x) \in \mathbf{R}$. Such an element $x \in X$ is called an $\mathbf{R}$-*minimal* element of X. (It is useful to think of $(u,v) \in \mathbf{R}$ as saying that "u $\mathbf{R}$-precedes v" or "u is $\mathbf{R}$-smaller than v", although $\mathbf{R}$ need not be an order or partial order.)

There is a formula saying that "$\mathbf{R}$ is well-founded on $\mathbf{A}$", namely:

$$(1)\quad \forall x\,\forall y\,\big(\varphi(x,y) \to (\psi(x) \wedge \psi(y))\big) \\ \wedge\ \forall X\,\big((X \neq \emptyset\ \wedge\ \forall x \in X\,\psi(x)) \to \exists x \in X\,\forall y \in X\,\neg\varphi(y,x)\big).$$

Example 2.2. The class relation $\mathbf{R}_{\in} = \{(x,y) : x \in y\}$ is well-founded on $\mathbf{V}$. This is exactly what the Axiom of Foundation says: for every nonempty set X there exists $x \in X$ such that

$$\text{no } y \in X \text{ satisfies } y \in x,$$

which is equivalent to $x \cap X = \emptyset$.

Example 2.3. The class relation $<$ on $\mathbf{On}$ is well-founded on $\mathbf{On}$. This is exactly the property needed to make the linear order $<$ on $\mathbf{On}$ a well-order. (The trichotomy law for $<$ implies that for any subset X of $\mathbf{On}$ and $x \in X$, the conditions "x is an $<$-minimal element of X" and "x is a least element of X" are equivalent.)

Definition 2.4. We say that $\mathbf{R}$ *is set-like on* $\mathbf{A}$ if $\mathbf{R} \subseteq \mathbf{A} \times \mathbf{A}$ and for every $x \in \mathbf{A}$, the class

$$\mathrm{pred}_{\mathbf{A},\mathbf{R}}(x) = \{y \in \mathbf{A} : (y, x) \in \mathbf{R}\}$$

is a set.

Again, there is a formula saying that "$\mathbf{R}$ is set-like on $\mathbf{A}$", namely:

$$\forall x\, \forall y\, \big(\varphi(x, y) \to (\psi(x) \wedge \psi(y))\big) \ \wedge\ \forall x\, \big(\psi(x) \to \exists z\, \forall y\, (y \in z \leftrightarrow \varphi(y, x))\big). \tag{2}$$

Example 2.2 (continued)**.** The class relation $\mathbf{R}_\in = \{(x, y) : x \in y\}$ is set-like on $\mathbf{V}$, because $\mathrm{pred}_{\mathbf{V},\mathbf{R}_\in}(x) = x$ for every $x \in \mathbf{V}$.

Example 2.3 (continued)**.** The class relation $<$ on $\mathbf{On}$ is set-like on $\mathbf{On}$ for the same reason: $\mathrm{pred}_{\mathbf{On},<}(x) = \{y \in \mathbf{On} : y < x\} = \{y \in \mathbf{On} : y \in x\} = x$ for every $x \in \mathbf{On}$.

Our goal is to prove a general recursion theorem, which describes how to 'construct' class functions defined on a class $\mathbf{A}$ by recursion, using a class relation $\mathbf{R}$ that is well-founded and set-like on $\mathbf{A}$.

For the proof we will need some properties of well-founded, set-like class relations, which we will discuss now. As before, $\mathbf{R}$ denotes an arbitrary class relation (defined by $\varphi(x, y)$) and $\mathbf{A}$ an arbitrary class (defined by $\psi(x)$).

Definition 2.5. The *transitive closure of* $\mathbf{R}$ is the class relation $\mathbf{R}^*$ defined by the formula[2]

$$\begin{aligned} \varphi^*(x, y) &\equiv \exists n \in \omega \backslash 1\ \exists f\ \big(f = \langle f_0, \dots, f_n\rangle \text{ is a function with domain } n +' 1 \\ &\qquad\qquad\qquad\qquad \wedge\ f_0 = y\ \wedge\ f_n = x\ \wedge\ \forall i < n\ \varphi(f_{i+'1}, f_i)\big) \\ &\equiv \exists n \in \omega \backslash 1\ \Phi(x, y, n). \end{aligned}$$

We will not need that $\mathbf{R}^*$ is transitive; a somewhat weaker transitivity property of $\mathbf{R}^*$ (see 2.6(ii)) will be sufficient. We will use the following notation: for any $x \in \mathbf{A}$, let

$$\mathrm{pred}'_{\mathbf{A},\mathbf{R}}(x) = \mathrm{pred}_{\mathbf{A},\mathbf{R}}(x) \cup \{x\}.$$

[2]This formula contains some abbreviations. For example, for any formula Φ, $\exists n \in \omega \backslash 1\ \Phi$ is an abbreviation for $\exists n\, (n \neq \emptyset\ \wedge\ \forall u\, (\iota(u) \to n \in u)\ \wedge\ \Phi)$, where $\iota(u)$ is the formula for "u is inductive".

Theorem 2.6. *Let* $\mathbf{R}$ *be a class relation, and let* $\mathbf{A}$ *be a class.*

(i) $\mathbf{R} \subseteq \mathbf{R}^*$.

(ii) *If* $\mathbf{R} \subseteq \mathbf{A} \times \mathbf{A}$ *and* $x \in \mathbf{A}$*, then for every pair* $(u, v) \in \mathbf{R}$ *such that* $v \in \mathrm{pred}'_{\mathbf{A},\mathbf{R}^*}(x)$ *we have that* $u \in \mathrm{pred}'_{\mathbf{A},\mathbf{R}^*}(x)$.

(iii) *If* $\mathbf{R}$ *is set-like on* $\mathbf{A}$*, then so is* $\mathbf{R}^*$.

(iv) *If* $\mathbf{R}$ *is well-founded and set-like on* $\mathbf{A}$*, then every nonempty sub*<u>*class*</u> *of* $\mathbf{A}$ *has an* $\mathbf{R}$*-minimal element.*

Idea of Proof of (iii). Fix $x \in \mathbf{A}$. We want to show that $\mathrm{pred}_{\mathbf{A},\mathbf{R}^*}(x)$ is a set. For each $n \in \omega \setminus 1$, consider the class

$$\mathbf{D}_n = \{y \in \mathbf{A} : \Phi(y, x, n)\}.$$

1. Use the Induction Principle for ω to show that $\mathbf{D}_n$ is a set for all $n \in \omega \setminus 1$:
   - ⋄ $\mathbf{D}_1 = \mathrm{pred}_{\mathbf{A},\mathbf{R}}(x)$.
   - ⋄ $\mathbf{D}_{n+'1} = \{y \in \mathbf{A} : \exists z \in \mathbf{D}_n \; y \in \mathrm{pred}_{\mathbf{A},\mathbf{R}}(z)\}$.
   - ⋄ Use the axiom of replacement to prove that if $\mathbf{D}_n$ is a set, then so is $\mathbf{D}_{n+'1}$, namely, $\mathbf{D}_{n+'1} = \bigcup\{\mathrm{pred}_{\mathbf{A},\mathbf{R}}(z) : z \in \mathbf{D}_n\}$.
2. Use replacement again to show that $\mathrm{pred}_{\mathbf{A},\mathbf{R}^*}(x) = \bigcup\{\mathbf{D}_n : n \in \omega \setminus 1\}$ is a set. □

We need one more fact before stating our general recursion theorem. Let $\mathbf{F}$ be a class relation defined by a formula $\theta(x, y)$. Recall that saying that $\mathbf{F}$ is a class function on $\mathbf{A}$ means that

$$\mathsf{ZFC} \vdash \forall x \left(\psi(x) \to \exists! y \, \theta(x, y)\right).$$

If this is the case, then for every sub<u>set</u> S of $\mathbf{A}$ the formula $x \in S \wedge \theta(x, y)$ defines a class function with domain S, which will be denoted by $\mathbf{F}{\upharpoonright}S$. By replacement, the range of this function is also a set, so by comprehension, the function

$$\mathbf{F}{\upharpoonright}S = \{(x, y) \in S \times \mathrm{rng}(\mathbf{F}{\upharpoonright}S) : \theta(x, y)\}$$

itself is a set.

Recursion Theorem 2.7. *Let* $\mathbf{A}$ *be a class, let* $\mathbf{R}$ *be a class relation that is well-founded and set-like on* $\mathbf{A}$*, and let* $\mathbf{G}$ *be a class function* $\mathbf{A} \times \mathbf{V} \to \mathbf{V}$*.*

(E) *There exists a class function* $\mathbf{F} \colon \mathbf{A} \to \mathbf{V}$ *such that*

$$\mathbf{F}(a) = \mathbf{G}\big(a, \mathbf{F}{\restriction}\mathrm{pred}_{\mathbf{A},\mathbf{R}}(a)\big) \qquad \textit{for all } a \in \mathbf{A}. \tag{3}$$

(U) *This class function is unique; that is, if* $\mathbf{F}$ *and* $\mathbf{F}'$ *are class functions* $\mathbf{A} \to \mathbf{V}$ *such that* (3) *holds for* $\mathbf{F}$ *and the same condition with* $\mathbf{F}$ *replaced by* $\mathbf{F}'$ *holds for* $\mathbf{F}'$*, then* $\mathbf{F} = \mathbf{F}'$*.*

To discuss the rigorous version of the Recursion Theorem, let $\mathbf{R}$ and $\mathbf{A}$ be defined by the formulas $\varphi(x, y)$ and $\psi(x)$ as before, and let $\mathbf{G}$ be defined by the formula $\chi(x, y, z)$. Our assumptions on $\mathbf{R}$ are described by the formulas (1) and (2), while our assumption on $\mathbf{G}$ is that ZFC proves

$$\forall x\, \forall y\, \forall z\, \big(\chi(x, y, z) \to \psi(x)\big) \;\wedge\; \forall x\, \forall y\, \big(\psi(x) \to \exists! z\, \chi(x, y, z)\big). \tag{4}$$

The statement in (3) is expressed by the sentence ρ_θ below, if θ is the formula that defines the class function $\mathbf{F}$ on $\mathbf{A}$:

$$\begin{aligned} \rho_\theta \equiv{}& \forall x\, \forall y\, \big(\theta(x, y) \to \psi(x)\big) \;\wedge\; \forall x\, \big(\psi(x) \to \exists! y\, \theta(x, y)\big) \\ &\wedge\; \forall x\, \Bigg(\psi(x) \to \exists y\, \Bigg(\theta(x, y) \;\wedge\; \exists f\, \Bigg(f \text{ is a function} \\ &\qquad \wedge\; \forall u\, \forall v\, \big((u, v) \in f \leftrightarrow (\varphi(u, x) \wedge \theta(u, v))\big) \;\wedge\; \chi(x, f, y)\Bigg)\Bigg)\Bigg). \end{aligned}$$

Thus, statement (E) of the Recursion Theorem asserts that there exists a formula $\theta(x, y)$ (involving φ, ψ, and χ) such that

$$\mathsf{ZF} \;\vdash\; [(1),\, (2),\, \text{and } (4)] \to \rho_\theta.$$

The proof will yield such a formula $\theta(x, y)$. The conclusion of part (U) of the Recursion Theorem is that for any other formula $\theta'(x, y)$,

$$\mathsf{ZF} \;\vdash\; [(1),\, (2),\, \text{and } (4)] \to \Big((\rho_\theta \wedge \rho_{\theta'}) \to \forall x\, \forall y\, \big(\theta(x, y) \leftrightarrow \theta'(x, y)\big)\Big).$$

Proof of the Recursion Theorem. (U) To prove the uniqueness statement, assume $\mathbf{F}\colon \mathbf{A} \to \mathbf{V}$ and $\mathbf{F}'\colon \mathbf{A} \to \mathbf{V}$ satisfy

$$\mathbf{F}(a) = \mathbf{G}\big(a, \mathbf{F}{\restriction}\mathrm{pred}_{\mathbf{A},\mathbf{R}}(a)\big) \quad \text{and} \quad \mathbf{F}'(a) = \mathbf{G}\big(a, \mathbf{F}'{\restriction}\mathrm{pred}_{\mathbf{A},\mathbf{R}}(a)\big) \qquad \text{for all } a \in \mathbf{A},$$

but $\mathbf{F} \neq \mathbf{F}'$. Then

$$\mathbf{X} = \{x \in \mathbf{A} : \mathbf{F}(x) \neq \mathbf{F}'(x)\}$$

is a nonempty subclass of $\mathbf{A}$, so by Theorem 2.6(iv), $\mathbf{X}$ has an $\mathbf{R}$-minimal element a. Hence, $\mathbf{F}(b) = \mathbf{F}'(b)$ for all $b \in \mathbf{A}$ with $(b, a) \in \mathbf{R}$, i.e., $\mathbf{F}{\restriction}\mathrm{pred}_{\mathbf{A},\mathbf{R}}(a) = \mathbf{F}'{\restriction}\mathrm{pred}_{\mathbf{A},\mathbf{R}}(a)$. It follows that

$$\mathbf{F}(a) = \mathbf{G}\big(a, \mathbf{F}{\restriction}\mathrm{pred}_{\mathbf{A},\mathbf{R}}(a)\big) = \mathbf{G}\big(a, \mathbf{F}'{\restriction}\mathrm{pred}_{\mathbf{A},\mathbf{R}}(a)\big) = \mathbf{F}'(a),$$

which contradicts $a \in \mathbf{X}$.

(E) Informally, the idea of the proof of the existence of $\mathbf{F}$ is the following: we show that there is a 'consistent' family (class!) of 'approximations' to $\mathbf{F}$ whose 'union' is $\mathbf{F}$.

More precisely, by an *approximation to* $\mathbf{F}$ we mean a (set) function f such that $\mathrm{dmn}(f) \subseteq \mathbf{A}$, and for every $a \in \mathrm{dmn}(f)$ we have

$$\mathrm{pred}_{\mathbf{A},\mathbf{R}}(a) \subseteq \mathrm{dmn}(f) \quad \text{and} \quad f(a) = \mathbf{G}(a, f{\restriction}\mathrm{pred}_{\mathbf{A},\mathbf{R}}(a)).$$

Formally, the class of all approximations f to $\mathbf{F}$ is defined by the following formula:

$$\begin{aligned}(5)\quad \mu(f) \equiv{}& f \text{ is a function} \wedge \forall x \big(x \in \mathrm{dmn}(f) \to \psi(x)\big) \wedge \\ & \forall a \Bigg(a \in \mathrm{dmn}(f) \to \exists g \Big(g \text{ is a function} \wedge \forall y \big(y \in \mathrm{dmn}(g) \leftrightarrow \varphi(y, a)\big) \\ &\qquad \wedge\ \forall y \in \mathrm{dmn}(g) \big(y \in \mathrm{dmn}(f) \wedge g(y) = f(y)\big) \wedge \chi(a, g, f(a))\Big)\Bigg).\end{aligned}$$

The class of all approximations to $\mathbf{F}$ has the following 'consistency' property:

Claim 2.8. *If f, f' are approximations to $\mathbf{F}$ and $a \in \mathrm{dmn}(f) \cap \mathrm{dmn}(f')$, then $f(a) = f'(a)$.*

This implies that 'the union of the approximations to $\mathbf{F}$', that is, the class relation defined by the formula

$$\theta(x, y) \equiv \exists f \big(\mu(f) \wedge x \in \mathrm{dmn}(f) \wedge f(x) = y\big),$$

is a class function $\mathbf{F}\colon \mathbf{A}' \to \mathbf{V}$ on some subclass $\mathbf{A}'$ of $\mathbf{A}$, namely, on 'the union of the domains of all approximations to $\mathbf{F}$'. Clearly, $\mathbf{A}'$ is defined by the formula

$$\psi'(x) \equiv \exists f \big(\mu(f) \wedge x \in \mathrm{dmn}(f)\big).$$

Next we show that $\mathbf{A}' = \mathbf{A}$, or equivalently, that every $x \in \mathbf{A}$ is in the domain of some approximation to $\mathbf{F}$. This is done in the following three steps:

Claim 2.9. *If f is an approximation to $\mathbf{F}$ and $x \in \mathrm{dmn}(f)$, then $\mathrm{pred}'_{\mathbf{A},\mathbf{R}^*}(x) \subseteq \mathrm{dmn}(f)$.*

Claim 2.10. *If f is an approximation to $\mathbf{F}$ and $x \in \mathrm{dmn}(f)$, then $f\restriction\mathrm{pred}'_{\mathbf{A},\mathbf{R}^*}(x)$ is an approximation to $\mathbf{F}$.*

Claim 2.11. *For each $x \in \mathbf{A}$ there exists an approximation f to $\mathbf{F}$ such that $x \in \mathrm{dmn}(f)$.*

Idea of Proof of Claim 2.11. Assume not. Then the class

$$\mathbf{X} = \{z \in \mathbf{A} : z \notin \mathrm{dmn}(f) \text{ holds for all approximations } f \text{ to } \mathbf{F}\}$$

is nonempty, so by Theorem 2.6(iv), $\mathbf{X}$ has an $\mathbf{R}$-minimal element x. Let us fix such an x.

1. For every $y \in \mathrm{pred}_{\mathbf{A},\mathbf{R}}(x)$,
   - ⋄ there exists an approximation f to $\mathbf{F}$ such that $y \in \mathrm{dmn}(f)$, and
   - ⋄ $\widehat{f_y} = f\restriction\mathrm{pred}'_{\mathbf{A},\mathbf{R}^*}(y)$ is the unique approximation to $\mathbf{F}$ with domain $\mathrm{pred}'_{\mathbf{A},\mathbf{R}^*}(y)$.
2. Consider the class $\mathcal{A} = \{\widehat{f_y} : y \in \mathrm{pred}_{\mathbf{A},\mathbf{R}}(x)\}$ (it is indeed a class!). Then
   - ⋄ $\mathcal{A}$ is a set, and
   - ⋄ $g = \bigcup \mathcal{A}$ is a function with domain
   $$\mathrm{dmn}(g) = \bigcup\{\mathrm{pred}'_{\mathbf{A},\mathbf{R}^*}(y) : y \in \mathrm{pred}_{\mathbf{A},\mathbf{R}}(x)\}(\subseteq \mathbf{A}).$$
3. g is an approximation to $\mathbf{F}$; i.e., for every $a \in \mathrm{dmn}(g)$,
   - ⋄ $\mathrm{pred}_{\mathbf{A},\mathbf{R}}(a) \subseteq \mathrm{dmn}(g)$, and
   - ⋄ $g(a) = \mathbf{G}\big(a, g\restriction\mathrm{pred}_{\mathbf{A},\mathbf{R}}(a)\big)$.
4. $h = g \cup \big\{\big(x, \mathbf{G}(x, g\restriction\mathrm{pred}_{\mathbf{A},\mathbf{R}}(x))\big)\big\}$ is an approximation to $\mathbf{F}$. Indeed,
   - ⋄ h is a function, and $x \in \mathrm{dmn}(h) \subseteq \mathbf{A}$;
   - ⋄ $\mathrm{pred}_{\mathbf{A},\mathbf{R}}(a) \subseteq \mathrm{dmn}(h)$ for all $a \in \mathrm{dmn}(h)$, and
   - ⋄ $h(a) = \mathbf{G}\big(a, h\restriction\mathrm{pred}_{\mathbf{A},\mathbf{R}}(a)\big)$ for all $a \in \mathrm{dmn}(h)$.

The fact that h is an approximation to $\mathbf{F}$ with $x \in \mathrm{dmn}(h)$, contradicts the choice of x. ⋄

Thus, $\mathbf{F}$ is a class function $\mathbf{A} \to \mathbf{V}$. We will now check that $\mathbf{F}$ also satisfies (3). Let $a \in \mathbf{A}$. By Claim 2.11, there is an approximation f to $\mathbf{F}$ such that $a \in \mathrm{dmn}(f)$. It follows from the definition of an approximation that $\mathrm{pred}_{\mathbf{A},\mathbf{R}}(a) \subseteq \mathrm{dmn}(f)$, hence Claim 2.8 and the definition of $\mathbf{F}$ imply that $\mathbf{F}\restriction\mathrm{pred}_{\mathbf{A},\mathbf{R}}(a) = f\restriction\mathrm{pred}_{\mathbf{A},\mathbf{R}}(a)$. Therefore,

$$\mathbf{F}(a) = f(a) = \mathbf{G}\big(a, f\restriction\mathrm{pred}_{\mathbf{A},\mathbf{R}}(a)\big) = \mathbf{G}\big(a, \mathbf{F}\restriction\mathrm{pred}_{\mathbf{A},\mathbf{R}}(a)\big).$$

Since $\mathbf{F}$ is defined by the formula $\theta(x, y)$, our proof above shows that $\theta(x, y)$ is indeed the formula we asserted to exist in part (E) of the Recursion Theorem.

The proof of the Recursion Theorem is complete. □

3. Transfinite Induction and Recursion

The transfinite induction principles for ordinals and for $\mathbf{On}$ follow from the following special case of Theorem 2.6(iv):

Corollary 3.1. *If $\mathbf{A}$ is an ordinal or $\mathbf{A} = \mathbf{On}$, then every nonempty subclass of $\mathbf{A}$ has a least element (with respect to $<$).*

3.2 and 3.3 below state two forms of the induction principle on ordinals or on $\mathbf{On}$.

Transfinite Induction Theorem 3.2. *Let $\mathbf{A}$ be an ordinal or $\mathbf{On}$. For every subclass $\mathbf{B}$ of $\mathbf{A}$, if*

- *for all $\alpha \in \mathbf{A}$ such that $\alpha \subseteq \mathbf{B}$ we have $\alpha \in \mathbf{B}$,*

then $\mathbf{B} = \mathbf{A}$.

In particular, for the cases $\mathbf{A} = \beta$ ($\in \mathbf{On}$) and $\mathbf{A} = \mathbf{On}$, this means the following:

($\diamond$) For every subset $X \subseteq \beta$, if
- for all $\alpha < \beta$ such that $\alpha \subseteq X$ we have $\alpha \in X$,

then $X = \beta$.

($\diamond\diamond$) For every class $\mathbf{B}$ of ordinals, if
- for all ordinals α such that $\alpha \subseteq \mathbf{B}$ we have $\alpha \in \mathbf{B}$,

then $\mathbf{B} = \mathbf{On}$.

Transfinite Induction Theorem 3.3. *Let $\mathbf{A}$ be an ordinal or $\mathbf{On}$. For every subclass $\mathbf{B}$ of $\mathbf{A}$, if*

- $0 \in \mathbf{B}$,
- *for all successor ordinals $\alpha +' 1 \in \mathbf{A}$ such that $\alpha \in \mathbf{B}$ we have $\alpha +' 1 \in \mathbf{B}$, and*
- *for all limit ordinals $\alpha \in \mathbf{A}$ such that $\alpha \subseteq \mathbf{B}$ we have $\alpha \in \mathbf{B}$,*

then $\mathbf{B} = \mathbf{A}$.

As before, it is easy to specialize the theorem for the cases when $\mathbf{A} = \beta$ ($\in \mathbf{On}$) or $\mathbf{A} = \mathbf{On}$.

The Transfinite Recursion Theorem below is an immediate consequence of the Recursion Theorem (2.7):

Transfinite Recursion Theorem 3.4. *Let $\mathbf{A}$ be an ordinal or $\mathbf{On}$. For every class function $\mathbf{G}\colon \mathbf{A} \times \mathbf{V} \to \mathbf{V}$ there exists a unique class function $\mathbf{F}\colon \mathbf{A} \to \mathbf{V}$ such that*

$$\mathbf{F}(\alpha) = \mathbf{G}(\alpha, \mathbf{F}\restriction\alpha) \quad \textit{for all } \alpha \in \mathbf{A}.$$

4. Well-Orders

Definition 4.1. A *partial order* (or *partially ordered set*) is a pair $(P, <)$ such that P is a set, $< \subseteq P \times P$, and $<$ is irreflexive and transitive. A partial order $(P, <)$ satisfying the trichotomy law is called a *linear order* (or *linearly ordered set*). A linear order $(P, <)$ such that $<$ is well-founded on P is called a *well-order* (or *well-ordered set*).

Recall that "$<$ is well-founded on P" means: for every nonempty $X \subseteq P$ there exists $x \in X$ such that x is $<$-minimal in X (i.e., $y \not< x$ for all $y \in X$). By trichotomy, such an element $x \in X$ is the *$<$-least element of* X (i.e., $y \geq x$ for all $y \in X$).

Example 4.2. $(\alpha, <)$ is a well-order for every ordinal α.

Definition 4.3. Let $(A, <)$ and $(B, \prec)$ be partial orders, and let $f \colon A \to B$ be a function. We say that f is *strictly increasing* (or *strict order preserving*) if for all $a_1, a_2 \in A$ with $a_1 < a_2$ we have $f(a_1) \prec f(a_2)$. We say that f is an *isomorphism from* $(A, <)$ *onto* $(B, \prec)$ (or an *isomorphism between* $(A, <)$ *and* $(B, \prec)$) if f is one-to-one, $\operatorname{rng}(f) = B$, and for all $a_1, a_2 \in A$ we have $a_1 < a_2$ if and only if $f(a_1) \prec f(a_2)$. We call two partial orders *isomorphic* if there is an isomorphism between them.

The main result we want to prove is the following theorem.

Theorem 4.4. *For every well-order* $(B, \prec)$ *there exists a unique ordinal* α *such that* $(\alpha, <)$ *is isomorphic to* $(B, \prec)$. *Moreover, there is a unique isomorphism from* $(\alpha, <)$ *onto* $(B, \prec)$.

Proof. Let $(B, \prec)$ be a well-order. The theorem is clearly true if $B = \emptyset$, therefore we will assume throughout the proof that B is nonempty.

To prove the existence of an ordinal α such that $(\alpha, <)$ is isomorphic to $(B, \prec)$ we define a class function $\mathbf{G} \colon \mathbf{On} \times \mathbf{V} \to \mathbf{V}$ as follows: for any $\gamma \in \mathbf{On}$ and $x \in \mathbf{V}$, let

$$\mathbf{G}(\gamma, x) = \begin{cases} \text{the } \prec\text{-least element of } B \setminus \operatorname{rng}(x) & \text{if } x \text{ is a function and } B \setminus \operatorname{rng}(x) \neq \emptyset, \\ B & \text{otherwise.} \end{cases}$$

Note that, since $B \notin B$, the equality $\mathbf{G}(\gamma, x) = B$ for a function x implies that $B \setminus \operatorname{rng}(x) = \emptyset$.

By the Transfinite Recursion Theorem there exists a class function $\mathbf{F}\colon \mathbf{On} \to \mathbf{V}$ such that

$$\mathbf{F}(\beta) = \mathbf{G}(\beta, \mathbf{F}{\restriction}\beta) \quad \text{for all } \beta \in \mathbf{On}.$$

The proof proceeds by showing that $\mathbf{F}$ has the following properties:

1. For $\beta < \gamma$ in $\mathbf{On}$,
   - if $\mathbf{F}(\beta) = B$, then $\mathbf{F}(\gamma) = B$; and
   - if $\mathbf{F}(\gamma) \neq B$ — and hence $\mathbf{F}(\beta) \neq B$ —, then $\mathbf{F}(\beta) \prec \mathbf{F}(\gamma)$.
2. There exists $\gamma \in \mathbf{On}$ such that $\mathbf{F}(\gamma) = B$.
3. For the least ordinal α such that $\mathbf{F}(\alpha) = B$, the function $\mathbf{F}{\restriction}\alpha$ maps α onto B.

By properties 1 and 3, $\mathbf{F}{\restriction}\alpha$ is strict order preserving from $(\alpha, <)$ onto $(B, \prec)$. As the following easy, but useful, claim shows, this is enough to conclude that $\mathbf{F}{\restriction}\alpha$ is, in fact, an isomorphism from $(\alpha, <)$ onto $(B, \prec)$.

Claim 4.5. *If $(A, <)$ and $(B, \prec)$ are linear orders, and $f\colon A \to B$ is strict order preserving, then for all $a_1, a_2 \in A$ we have $a_1 < a_2$ if and only if $f(a_1) \prec f(a_2)$.*

The proof of the uniqueness of α and the isomorphism between $(\alpha, <)$ and $(B, \prec)$ follow from Claims 4.7 and 4.8 below, both of which rely on the following easy, but useful fact.

Claim 4.6. *If $(A, <)$ is a well-order and $f\colon A \to A$ is strict order preserving, then $x \leq f(x)$ for all $x \in A$.*

Claim 4.7. *If α and β are different ordinals, then $(\alpha, <)$ and $(\beta, <)$ are not isomorphic.*

Claim 4.8. *If $(A, <)$ and $(B, \prec)$ are isomorphic well-orders, then there is a unique isomorphism from $(A, <)$ onto $(B, \prec)$.*

$\square$

5. Ordinal Class Functions and Ordinal Arithmetic

Definition 5.1. By an *ordinal class function* we mean a class function $\mathbf{F} \subseteq \mathbf{On} \times \mathbf{On}$ whose domain is either an ordinal or the class $\mathbf{On}$. We say that an ordinal class function $\mathbf{F}$ with domain $\mathbf{A}$ is

- *strictly increasing* (or *strict order preserving*) if for all $\alpha, \beta \in \mathbf{A}$ with $\alpha < \beta$ we have $\mathbf{F}(\alpha) < \mathbf{F}(\beta)$;
- *continuous* if for every limit ordinal $\alpha \in \mathbf{A}$ we have $\mathbf{F}(\alpha) = \bigcup_{\beta<\alpha} \mathbf{F}(\beta)$;
- *normal* if it is continuous and strict order preserving.

Some basic properties of ordinal class functions satisfying some of these properties are summarized in the following theorem. The first statement of the theorem is the analog of Claim 4.6 for ordinal class functions.

Theorem 5.2. *Let $\mathbf{F}$ be an ordinal class function with domain $\mathbf{A}$, and let $\mathbf{G}$ be an ordinal class function with domain $\mathbf{B}$.*

(i) *If $\mathbf{F}$ is strict order preserving, then $\alpha \leq \mathbf{F}(\alpha)$ for all $\alpha \in \mathbf{A}$.*
(ii) *If $\mathbf{F}$ is continuous and for every successor ordinal $\alpha +' 1 \in \mathbf{A}$ we have $\mathbf{F}(\alpha) < \mathbf{F}(\alpha +' 1)$, then $\mathbf{F}$ is strict order preserving (hence normal).*
(iii) *If $\mathbf{F}$ is normal, then $\mathbf{F}(\alpha)$ is a limit ordinal for every limit ordinal $\alpha \in \mathbf{A}$.*
(iv) *If $\mathbf{F}$ and $\mathbf{G}$ are normal and the range of $\mathbf{F}$ is contained in $\mathbf{B}$, then $\mathbf{G} \circ \mathbf{F}$ is normal.*

The general Recursion Theorem 2.7 is used to define addition, multiplication, and exponentiation for ordinals.

Theorem 5.3. *There exists a unique class function* $+\colon \mathbf{On}\times\mathbf{On}\to\mathbf{On}$ *such that the following conditions hold for any* $\alpha,\beta\in\mathbf{On}$*:*

- $\alpha+0=\alpha$*;*
- $\alpha+\beta=(\alpha+\gamma)+'1$ *if* $\beta=\gamma+'1$ *is a successor ordinal;*
- $\alpha+\beta=\bigcup_{\gamma<\beta}(\alpha+\gamma)$ *if* β *is a limit ordinal.*

Idea of Proof. For the existence of $+$, let $\mathbf{A}=\mathbf{On}\times\mathbf{On}$ and

$$\mathbf{R}=\big\{\big((\alpha,\beta),(\alpha,\gamma)\big):\alpha,\beta,\gamma\in\mathbf{On}\text{ with }\beta<\gamma\big\}.$$

It is easy to see that $\mathbf{R}$ is well-founded and set-like on $\mathbf{A}$. Now let $\mathbf{G}\colon\mathbf{A}\times\mathbf{V}\to\mathbf{V}$ be the class function defined for any $(\alpha,\beta)\in\mathbf{A}$ $(=\mathbf{On}\times\mathbf{On})$ and $x\in\mathbf{V}$ by

$$\mathbf{G}\big((\alpha,\beta),x\big)=\begin{cases}\alpha & \text{if }\beta=0,\\ x(\alpha,\gamma)\cup\{x(\alpha,\gamma)\} & \text{if }\beta=\gamma+'1\text{ is a successor ordinal}\\ & \text{and }x\text{ is a function with domain }\{\alpha\}\times\beta,\\ \bigcup_{\gamma<\beta}x(\alpha,\gamma) & \text{if }\beta\text{ is a limit ordinal}\\ & \text{and }x\text{ is a function with domain }\{\alpha\}\times\beta,\\ \emptyset & \text{otherwise.}\end{cases}$$

By the Recursion Theorem there exists a unique class function $\mathbf{F}\colon\mathbf{A}\to\mathbf{V}$ such that

$$\begin{aligned}\mathbf{F}(\alpha,\beta)&=\mathbf{G}\big((\alpha,\beta),\mathbf{F}\restriction\mathrm{pred}_{\mathbf{A},\mathbf{R}}((\alpha,\beta))\big)\\ &=\mathbf{G}\big((\alpha,\beta),\mathbf{F}\restriction(\{\alpha\}\times\beta)\big)\qquad\text{for all }\alpha,\beta\in\mathbf{On}.\end{aligned}$$

Denoting $\mathbf{F}(\alpha,\beta)$ by $\alpha+\beta$ we get that $+$ has the stated properties.

For each $\alpha\in\mathbf{On}$, the uniqueness of the class function $\alpha+\_\colon\mathbf{On}\to\mathbf{On}$ satisfying the conditions listed in the theorem follows by transfinite induction (say, by 3.3). This proves the uniqueness of $+$. ☐

Notice that,

$$\alpha+'1=\alpha+1\quad\text{for all }\alpha\in\mathbf{On},$$

because 1 is the successor of 0, i.e., $1=0+'1$, and hence by the properties of $+$ we have that $\alpha+1=\alpha+(0+'1)=(\alpha+0)+'1=\alpha+'1$. Therefore, from now on, we will write $\alpha+1$ for the successor of an ordinal α.

The basic properties of ordinal addition are summarized in the next theorem.

Theorem 5.4. *The following hold for arbitrary ordinals α, β, γ:*

(i) *If $\alpha, \beta \in \omega$, then $\alpha + \beta \in \omega$.*

(ii) *The ordinal class function $\mathbf{F}_{\alpha+}\colon \mathbf{On} \to \mathbf{On}$, $\delta \mapsto \alpha + \delta$ has the following properties:*

$\quad$(ii)$_1$ *$\mathbf{F}_{\alpha+}$ is normal;*

$\quad$(ii)$_2$ *$\mathbf{F}_{\alpha+}$ is one-to-one and maps onto $\mathbf{On} \setminus \alpha = \{\beta \in \mathbf{On} : \alpha \le \beta\}$;*

$\quad$(ii)$_3$ *we have $\delta \le \mathbf{F}_{\alpha+}(\delta) = \alpha + \delta$ for all $\delta \in \mathbf{On}$.*

(iii) *$0 + \alpha = \alpha$.*

(iv) *$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.*

(v) *If $\beta \le \gamma$ then $\beta + \alpha \le \gamma + \alpha$.*
(I.e., the ordinal class function $\mathbf{F}_{+\alpha}\colon \mathbf{On} \to \mathbf{On}$, $\delta \mapsto \delta + \alpha$ is 'weakly increasing' or '(weak) order preserving'.)

Idea of Proof. (i) Fix α and use the induction theorem for ω (Theorem 1.10).

(ii): For (ii)$_1$ use the defining properties of $+$ and Theorem 5.2(ii). This implies, in particular, that $\mathbf{F}_{\alpha+}$ is one-to-one. The remaining statement in (ii)$_2$ is equivalent to saying that for any ordinals α and β we have $\alpha \le \beta$ if and only if $\alpha + \delta = \beta$ for some $\delta \in \mathbf{On}$. This can be proved by transfinite induction, fixing α. (Induct on β in the forward direction, and on δ in the backward direction.) (ii)$_3$ follows from (ii)$_1$, using Theorem 5.2(i).

(iii) Use transfinite induction on α.

(iv) Fix α, β, and apply transfinite induction on γ, using the class functions $\mathbf{F}_{\alpha+}$, $\mathbf{F}_{\beta+}$, $\mathbf{F}_{(\alpha+\beta)+}$ from (ii) and their compositions (cf. Theorem 5.2(iv)).

(v) Fix β, γ, and use transfinite induction on α. (Alternatively: Use (ii)$_2$ to conclude that $\gamma = \beta + \varepsilon$ for some $\varepsilon \in \mathbf{On}$, and combine results from (ii) and (iv).) $\square$

Note that ordinal addition is not commutative. For example, $\omega + 1 > \omega$, because $\omega + 1$ is the successor of ω, while $1 + \omega = \omega$, because by the defining properties of $+$ we have $1 + \omega = \bigcup_{n<\omega}(1 + n) = \omega$.

Essentially the same example also shows that the ordinal class function $\mathbf{F}_{+\alpha}$ from Theorem 5.4(v) is not continuous, e.g., for $\alpha = 1$. Indeed, $\mathbf{F}_{+1}(\omega) = \omega + 1$, while $\bigcup_{n<\omega} \mathbf{F}_{+1}(n) = \bigcup_{n<\omega}(n + 1) = \omega$.

Similarly to ordinal addition, ordinal multiplication is defined by recursion, as the unique class function described in the next theorem.

Theorem 5.5. *There exists a unique class function* $\cdot\colon \mathbf{On} \times \mathbf{On} \to \mathbf{On}$ *such that the following conditions hold for any* $\alpha, \beta \in \mathbf{On}$*:*

- $\alpha \cdot 0 = 0$*;*
- $\alpha \cdot \beta = \alpha \cdot \gamma + \alpha$ *if* $\beta = \gamma + 1$ *is a successor ordinal;*
- $\alpha \cdot \beta = \bigcup_{\gamma<\beta}(\alpha \cdot \gamma)$ *if* β *is a limit ordinal.*

Some basic properties of ordinal multiplication are summarized in the next two theorems.

Theorem 5.6. *The following hold for arbitrary ordinals* α, β, γ*:*

(i) *If* $\alpha, \beta \in \omega$*, then* $\alpha \cdot \beta \in \omega$*.*

(ii) *The ordinal class function* $\mathbf{F}_{\alpha\cdot}\colon \mathbf{On} \to \mathbf{On}$, $\delta \mapsto \alpha \cdot \delta$ *has the following properties:*

(ii)$_1$ $\mathbf{F}_{\alpha\cdot}$ *is normal if* $\alpha \neq 0$*;*

(ii)$_2$ $\mathbf{F}_{\alpha\cdot}(1) = \alpha \cdot 1 = \alpha$ *and* $\mathbf{F}_{\alpha\cdot}(2) = \alpha \cdot 2 = \alpha + \alpha$*;*

(ii)$_3$ $\alpha \cdot (\beta + \gamma) = \mathbf{F}_{\alpha\cdot}(\beta + \gamma) = \mathbf{F}_{\alpha\cdot}(\beta) + \mathbf{F}_{\alpha\cdot}(\gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$*;*

(ii)$_4$ *if* $\alpha \neq 0$*, then* $\delta \leq \mathbf{F}_{\alpha\cdot}(\delta) = \alpha \cdot \delta$ *for all* $\delta \in \mathbf{On}$*.*

(iii) $0 \cdot \alpha = 0$*.*

(iv) $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$*.*

(v) *If* $\beta \leq \gamma$ *then* $\beta \cdot \alpha \leq \gamma \cdot \alpha$*.*
(I.e., the ordinal class function $\mathbf{F}_{\cdot\alpha}\colon \mathbf{On} \to \mathbf{On}$, $\delta \mapsto \delta \cdot \alpha$ *is 'weakly increasing' or '(weak) order preserving'.)*

Theorem 5.7. (Division Algorithm) *For arbitrary ordinals* α *and* β *such that* $\beta \neq 0$*, there exist unique ordinals* ξ *and* η *such that* $\alpha = \beta \cdot \xi + \eta$ *and* $\eta < \beta$*.*

For the definition and the properties of ordinal exponentiation, see Section 9 of [1].[3]

[3] [1] *Lectures on Set Theory* by J. Donald Monk.