Topological algebra based on sorts and properties as free and cofree universes

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BLAST 2010
CU Boulder
June 2

Well-structured categories (locally small)

Motivation A well-structured category should satisfy WS1-5.

WS1. The objects $U, V, \ldots$ of $\mathcal{C}$ are organized as **universes** equipped with **algebraic** or spatial structure and **topological** or localic structure listing possible paintings or (abstract) worlds.

WS2. Algebra is organized in terms of **elements** or points of universes classified by **sort** $s, t, \ldots$. Topology is organized in terms of **states** or opens in universes each interpreting some **property** p of the elements. We extend topology to permit more than one property.

WS3. Universal algebra is furnished with **operations** $f : s \to t$ between sorts as functions acting on elements. Universal topology is furnished with **dependencies** $d : p \to q$ between properties as functions on states.

WS4. Operations and dependencies are governed by equations.

WS5. There are enough free and cofree universes to contain all terms.

Examples

Grph 2 sorts: *vertex*, *edge*. $s, t : E \to V$. Discrete (no properties).

Grp ω sorts: G^i, $i < \omega$. $e : G^0 \to G^1$, $m : G^2 \to G^1$. Discrete.

Set 1 sort: *element*. Discrete.

Top 1 sort: *point*. 1 property: *in* (open set).

Vct_k 1 sort: *vector*. 1 property: *value* (of functional, $\in k$) (cf. Top)

Chu(Set, K) As for Top.

Ab_{LC} (locally compact abelian groups) As for Top. $K = \mathbb{R}/\mathbb{Z}$.

κ-*Locales* No sorts (pointless), κ properties K_α $\alpha < \kappa$.
$\bigvee_\alpha : L^\alpha \to L$, $\wedge_n : L^n \to L^1$, $n < \omega$.

$\mathcal{C}$ *inconsistent:* no sorts or properties. All universes empty.

Simple ontology of domestic pets: 3 sorts: *cat*, *dog*, *mammal*
2 operations: $is_{cm} : cat \to mammal$, $is_{dm} : dog \to mammal$
3 properties: *weight*, *color*, *hue*
1 dependency: *hue-of*: $color \to hue$.

Outline

1. PROGRAM: topoalgebra = Yoneda + duality. Simple, general.
2. PRETAC: Pretopoalgebraic category $(\mathcal{C}, s, t, \ldots, p, q, \ldots)$
   Algebra $s, t, \ldots \in \mathrm{ob}(\mathcal{C})$ are **sorts**
   Topology $p, q, \ldots \in \mathrm{ob}(\mathcal{C})$ are **properties** (attributes)
3. DEFINITIONS:
   element (point) $a^* : s \to U$, forming **carrier** U_s
   state (open) $x : U \to p$ forming **cocarrier** U^p
   map $h : U \to V$, acts on left on elements, h^* on right on opens
   quale k forming **field** K_s^p, **inner product** $U_s \times U^p \to K_s^p$
   operation $f^* : t \to s$, **dependency** $d : p \to q$
   universe U, **free** univ. $F_s = s$, **cofree** univ. $K^p = p$. $F_s^p = K_s^p$
4. THEOREM: Every map is homomorphic and continuous
5. TAC = **complete dense extension** of a pretac.
6. ARITY = functorial sorts (s^2) & properties (algebra, coalgebra,...)