Geodesic spaces : momentum
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Groups : symmetry

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1. Reprise of the relevant bits of my BLAST '08/9 talks

Goal: express each of the five postulates of Book I of Euclid's *Elements* equationally.

Their bilinear content is confined to the 3rd and 4th postulates, concerning respectively circles and right angles.

Bilinearity is equational. ✓

But those equations depend on numbers, which Book I outlawed. The obvious trick of identifying the Euclidean line with the underlying field would appear to inevitably lose information in a way that prevents the square of the line from being a Euclidean space.

Absent bilinearity we have only affine spaces.

Question: Can each of postulates 1, 2, and 5 be written equationally, observing the proscription on numbers, so that together they define a variety of affine spaces over some field k?

Answer: Yes, for k each of $\mathbb{Q}$ and $\mathbb{Q}[\mathbf{i}]$ (complex rationals)

2. Approach

1. We defined a variety **Grv** of "groves" with a binary operation ab denoting the point to which segment AB must be produced to double its length, interpretable in **Ab** as $ab = 2b - a$. Writing abc for $(ab)c$, we expressed Postulate 2 as $aa = abb = a$,
while Postulate 5 became $ab(cd) = ac(bd)$.

2. We equipped **Grv** with ω many commutative but non-associative n-ary centroid operations $a_1 \oplus a_2 \oplus \ldots \oplus a_n$.
We wrote Postulate 1 as two equations

$$\begin{aligned} a_1 \oplus \ldots \oplus a_{n-1} \oplus ((a_1 \oplus \ldots \oplus a_{n-1}) \xrightarrow{n} b) &= b \\ (a_1 \oplus \ldots \oplus a_{n-1}) \xrightarrow{n} (a_1 \oplus \ldots \oplus a_n) &= a_n, \end{aligned}$$

for each centroid operation in terms of ab ($a \xrightarrow{4} b = abab$ etc.)
We showed that the resulting variety is equivalent to $\mathbf{Aff}_{\mathbb{Q}}$.

3. We extended $\mathbb{Q}$ to $\mathbb{Q}[\mathbf{i}]$ with a binary operation $a \cdot b$ denoting b rotated 90 degrees about a.

End of reprise.

3. This talk; Geodesic spaces

At FMCS (Vancouver May 2009) Pieter Hofstra asked:

Can non-Euclidean geometry be treated analogously?

My answer (weeks later): weaken Postulate 5 to right distributivity,

$$abc = ac(bc).$$

Thinking of ba, a, b, ab, etc. as points evenly spaced along a geodesic γ, right distributivity expresses a symmetry of γ about an arbitrary point c, namely that the inversion γc in $c = \ldots, bac, ac, bc, abc, \ldots$ is itself a geodesic, namely $\ldots, bc(ac), ac, bc, ac(bc), \ldots$.

These algebras have sometimes been identified with quandles as used to algebraicize knot theory. This is wrong because the quandle operations interpreted in **Grp** are $b^{-1}ab$ and bab^{-1}, which collapse in **Ab** to $ab = a$, whereas the above is $ba^{-1}b$ which is very useful in **Ab**.

4. Geodesic theory

A geodesic space or **geode** is an algebraic structure with a binary operation $x \to y$, or xy, of **extension** (with xyz for $(xy)z$) satisfying

G0 $xx = x$ **G1** $xyy = x$ **G2** $xyz = xz(yz)$

Geometrically, segment A_0A_1 is *extended to* $A_2 = A_0 \to A_1$ by producing A_0A_1 to twice its length: $|A_0A_2| = 2|A_0A_1|$.

Examples

Symmetric spaces: Affine, hyperbolic, elliptic, etc.

Groups: Interpret $x \to y$ as $yx^{-1}y$ (abelian groups: $2y - x$)

Number systems: Integers, rationals, reals, complex numbers, etc.

Combinatorial structures: sets, dice, etc.

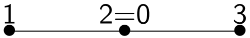
5. Geodesics

A **discrete geodesic** $\gamma(A_0, A_1)$ is a subspace generated by A_0, A_1.

A **geodesic** in S is a directed union of discrete geodesics in S.

Examples: $\mathbb{Z}$, $\mathbb{Z}_n$, $\mathbb{Q}$, $\mathbb{Q}/\mathbb{Z}$, $\mathbb{E}$ (§11). Not $\mathbb{R}$ (not fully represented).

Geodesics properly generalize cyclic groups.

Example: $\mathbb{E} = \mathbb{Z}_4/\{0 = 2\}$. 

S is **torsion-free** when every finite geodesic in S is a point.

The **connected components** of $\gamma(A_0, A_1)$ are $\ldots, A_{-2}, A_0, A_2, \ldots$ and $\ldots A_{-1}, A_1, A_3, \ldots$. These become one component just when $A_0 = A_{2n+1}$ for some n, as with $\mathbb{Z}_3$, $\mathbb{Z}_5$, etc.

The category Gsp

Geode homomorphism: a map $h : S \to T$ s.t. $h(xy) = h(x)h(y)$.

Denote by **Gsp** the category of geodes and their homomorphisms.

6. Sets

Theorem 1. *For any space S, the following are equivalent.*

(i) $\gamma(A, B) = \{A, B\}$ *for all* $A, B \in S$ (cf. $\gamma(N, S)$, N&S poles).

(ii) *The connected components of S are its points.*

(iii) $xy = x$ *for all* $x, y \in S$.

A **set** is a geode S with any (hence all) of those properties.

Define $U_{\mathbf{SetGsp}} : \mathbf{Set} \to \mathbf{Gsp}$ as $U_{\mathbf{SetGsp}}(X) = (X, \pi_1^2)$, i.e. $xy \overset{\text{def}}{=} x$.
Left adjoint $F_{\mathbf{GspSet}}(S) =$ the set of connected components of S.

Cf. $\mathcal{D} : \mathbf{Set} \to \mathbf{Top}$ where $\mathcal{D}(X) = (X, 2^X)$, a discrete space.

These embed **Set** fully in **Top** (**Pos**, **Grph**, **Cat**, etc.) and **Gsp**.
In **Top** etc. the embedding $\mathcal{D}$ preserves colimits.
In **Gsp** the (reflective) embedding $U_{\mathbf{SetGsp}}$ preserves limits!

In **Set**, $1 + 1 = 2$ and $2^{\aleph_0} = \beth_1$ (discrete continuum).
In **Top**, $1 + 1 = 2$ but $2^{\aleph_0} =$ Cantor space, not discrete.
In **Gsp**, $2^{\aleph_0} = \beth_1$, discrete (!), but $1 + 1 = \mathbb{Z}$, a homogeneous (no origin) geodesic with two connected components.

7. Normal form terms and free spaces

A **normal form** geodesic algebra term over a set X of variables is one with no parentheses or stuttering, namely a finite nonempty word $x_1 x_2 \ldots x_n$ over alphabet X with no consecutive repetitions.

Theorem 2. All terms are reducible to normal form using G0-G2. (G2 removes parentheses while G1 and G0 remove repetitions.)

Theorem 3. The normal form terms over X form a geode.

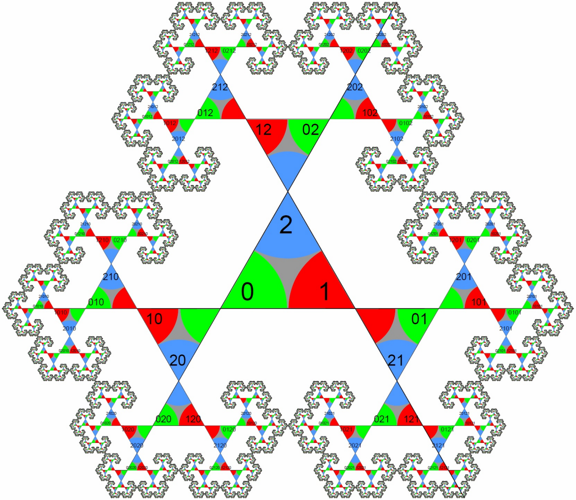
Denote this space by $F(X)$, the **free space** on X consisting of the "X-ary" operations. $F(\{\}) = \mathbf{0}$ (initial), $F(\{0\}) = \mathbf{1}$ (final).

$F(\{0,1\}) = \mathbf{1} + \mathbf{1}$ has two connected components $\mathbf{0}\alpha$ and $\mathbf{1}\alpha$.
It is an infinite discrete geodesic $\gamma(0,1) = \{0 \xrightarrow{n} 1\} =$

$$\mathbb{Z} \quad = \quad \ldots, 1010, 010, 10, 0, 1, 01, 101, 0101, \ldots$$

Call this *geodesimal notation*, tally notation with sign and parity bits.
Geodesimal operations: $x \xrightarrow{3} y = yxy$, $x \xrightarrow{-3} y = yxyx$, etc.

8. The free space 1+1+1. 3 connected components $0\alpha, 1\alpha, 2\alpha$

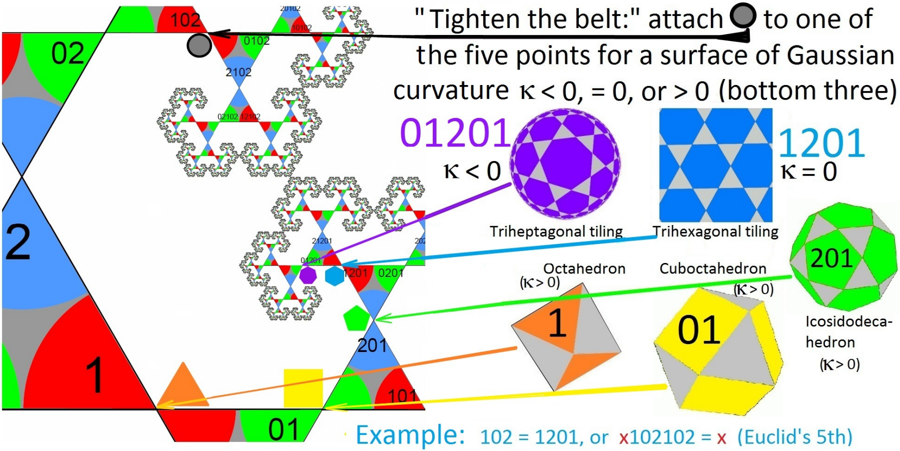


All points out to ∞ shown. Curvature κ undefined $(-\infty)$.

Triangles congruent by defn. but $\angle$, $\angle$, and $\angle$ incomparable.

$\exists$ disjoint inclined geodesics: $\gamma(101, 201) \cap \gamma(102, 202) = \emptyset$ (barely!)

9. The curvature hierarchy



All spaces (including $1 + 1 + 1$ itself) homogeneous.
Not shown: Sets ($xy = x$, §3), Dice ($xyxy = x$, §11).

10. Dice and subdirect irreducibles of Grv

The **edge** $\mathbb{E} = \mathbb{E}_3 = \{1, 0 = 2, 3\}$ is the unique geodesic with an odd number of points and two connected components.

- $\mathbb{E}_3 = \mathbb{Z}_4/\{0 = 2\}$
- $\mathbb{E}_6 = \mathbb{Z}_8/\{0 = 4, 2 = 6\}$
- $\mathbb{E}_{12} = \mathbb{Z}_{16}/\{0 = 8, 2 = 10, 4 = 12, 6 = 14\}$, etc.

Ab and **Grv** have the same SI's (subdirect irreducibles), namely $\mathbb{Z}_{p^n}$, $n \leq \infty$, as groves, except for $p = 2$ when $\mathbb{Z}_{4.2^n}$ is replaced by $\mathbb{E}_{3.2^n}$ in **Grv**. ($\mathbb{Z}_{p^\infty}$ is the Prüfer p-group = the direct limit of the inclusion $\mathbb{Z}_{p^0} \subseteq \mathbb{Z}_{p^1} \subseteq \mathbb{Z}_{p^2} \subseteq \ldots$.) Key fact: $\mathbb{Z}_4$ is a subdirect product of $\mathbb{E}$'s.

$\mathbb{E} \in \mathcal{V}$ iff $\mathbb{Z}_4 \in \mathcal{V}$ for all varieties $\mathcal{V} \subseteq$ **Gsp**.

A **die** is a subspace of $\mathbb{E}^n$, $n \leq \infty$. Equivalently, a model of $xx = xyy = x$, $xyxy{=}x$.

Dice $= HSP(\mathbb{Z}_4) = SP(\mathbb{E}) \subset$ **Grv**.

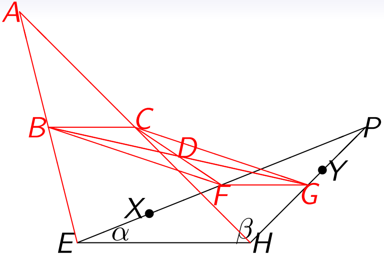
11. The geodesic neighborhood

Operations: xy $\qquad$ $[yxz]$ $\qquad$ xy, x^{-1}, e

$$
\begin{tikzcd}[column sep=large, row sep=large]
\mathbf{Gsp} \arrow[r, shift right, "xyxz"'] \arrow[dd, bend right=60, "x\ (\pi^2_1)"'] & \mathbf{Heap} \arrow[l, shift right, "{[yxy]}"'] & \mathbf{Grp} \arrow[l, "yx^{-1}z"'] & \\
\mathbf{Grv} \arrow[u, "\text{G3: } xywz = xzwy"] \arrow[r, shift right, "xyxz"'] & \mathbf{Schar} \arrow[u, "{[yxz]=[zxy]}"'] \arrow[l, shift right, "{[yxy]}"'] & \mathbf{Ab} \arrow[u, "xy = yx"'] \arrow[l, "y-x+z"'] & \mathbf{Ring} \arrow[l, "x-y"'] \\
\mathbf{Set} \arrow[u, "xy = x"'] & \mathbf{Cube} \arrow[u, "{[yxy]=x}"'] \arrow[l, "x\ (\pi^3_2)"'] & \mathbf{Cube}_* \arrow[u, "x+x=0"'] \arrow[l, "y+x+z"'] & \mathbf{Bool} \arrow[u, "x^2 = x"'] \arrow[l, "x+y"']
\end{tikzcd}
$$

Every path in this commutative diagram denotes a forgetful functor, hence one with a left adjoint. Vertical arrows *forget* the indicated *equation*, horizontal arrows *interpret* the blue *operation* above as the arrow's label. E.g. the left adjoint of the functor $U_{\mathbf{AbGrp}} : \mathbf{Ab} \to \mathbf{Grp}$ is abelianization, the arrow to **Schar** from **Ab** interprets **Schar**'s $[yxz]$ as $y - x + z$ in **Ab**, the left adjoint of the functor $U_{\mathbf{SetGsp}} : \mathbf{Set} \to \mathbf{Gsp}$ gives the set $F_{\mathbf{GspSet}}(S)$ of connected components of S, and so on.

12. Groves: Grv = Gsp + G3. Euclid's 5th postulate



Euclid's fifth or parallel postulate: EX and HY, when inclined inwards, meet when *produced*. Euclid: "inclined" $= \alpha + \beta < 180^o$.

Our inclination condition: a *witness triangle* ΔAEH with parallelogram $BCGF$ (centroid D) s.t B, C at midpoints of AE, AH.

Our 5th postulate: EF and HG, when obtained by extending the four sides of the skew quadrilateral $ABDC$, meet when *extended*.

$$\begin{array}{ccccccc} A \to B \to (C \to D) & = & A \to C \to (B \to D) \\ E \quad \to \quad F & = & H \quad \to \quad G \end{array} \qquad \textbf{(G3)}$$

G3 $xy(zw) = xz(yw) \mid xywz = xzwy \mid xywzywz = x \mid x102102 = x$