

# Topological algebra based on sorts and properties as free and cofree universes

Vaughan Pratt

Stanford University

BLAST 2010  
CU Boulder  
June 2

# MOTIVATION

A well-structured category  $\mathcal{C}$  should satisfy WS1-5.

WS1. The objects  $U, V, \dots$  of  $\mathcal{C}$  are organized as **universes** equipped with **algebraic** or spatial structure and **topological** or localic structure listing possible paintings or (abstract) worlds.

WS2. Algebra is organized in terms of **elements** or points of universes classified by **sort**  $s, t, \dots$ . Topology is organized in terms of **states** or opens in universes each interpreting some **property**  $p$  of the elements. We generalize topology to permit more than one property.

WS3. Universal algebra is furnished with **operations**  $f : s \rightarrow t$  between sorts as functions acting on elements. Universal topology is furnished with **dependencies**  $d : p \rightarrow q$  between properties as functions on states, as yet another generalization.

WS4. Operations and dependencies are governed by **equations**.

WS5.  $\exists$  enough **free** and **cofree** universes to contain all terms.

# EXAMPLES

**Grph** 2 sorts: *vertex*, *edge*.  $s, t : E \rightarrow V$ . Discrete (no properties).

**Grp**  $\omega$  sorts:  $G^i, i < \omega$ .  $e : G^0 \rightarrow G^1, m : G^2 \rightarrow G^1$ . Discrete.

**Set** 1 sort: *element*. Discrete.

**Top**  $s = \{1\}, p = \text{Sierpinski sp.}, 3$  dependencies.

**Vct<sub>k</sub>**  $\omega$  sorts and properties  $s^n = p^n = k^n$ . Ops & deps l.t.'s

**Chu(Set, K)** As for **Top**  $s = \{1\}, p = K$ , both rigid.

**Ab<sub>LC</sub>** (locally compact abelian groups) As for **Top**.  $K = \mathbb{R}/\mathbb{Z}$ .

**$\kappa$ -Locales**  $\kappa$ -Frm<sup>OP</sup>: No sorts (pointless),  $\kappa$  properties  $K^\alpha \alpha < \kappa$ .

$\bigvee_\alpha : L^\alpha \rightarrow L$  for  $\alpha < \kappa, \bigwedge_n : L^n \rightarrow L^1$  for  $n < \omega$ .

**Stone spaces** Locales with  $\kappa$  and Frm replaced by  $\omega$  and Bool.

**Simple ontology of domestic pets:** 3 sorts: *cat*, *dog*, *mammal*

2 operations:  $is_{cm} : cat \rightarrow mammal, is_{dm} : dog \rightarrow mammal$

3 properties: *weight*, *color*, *hue*

1 dependency: *hue-of*: *color*  $\rightarrow$  *hue*.

# OUTLINE

1. PROGRAM: topoalgebra = Yoneda + duality. Simple, general.
2. PRETAC: Pretopoalgebraic category  $\mathcal{C} = (C, S, P)$   
*Algebra*  $S \subseteq \text{ob}(C)$  consists of **sorts**  $s, t, \dots$   
*Topology*  $P \subseteq \text{ob}(C)$  consists of **properties** (attributes)  $p, q, \dots$
3. DEFINITIONS:  
**element** (point)  $a : s \rightarrow U$ , forming **carrier**  $U_s = \text{Hom}(s, U)$ ,  $s \in S$   
**state** (open)  $x : U \rightarrow p$  forming **cocarrier**  $U^p = \text{Hom}(U, p)$ ,  $p \in P$   
**map**  $h : U \rightarrow V$ , acts on left on elements and on right on states  
**operation**  $f^* : t \rightarrow s$ , **dependency**  $d : p \rightarrow q$   
**qualia**  $k : s \rightarrow p$  form a **field**  $K_s^p$ . **Inner product**  $U_s \times U^p \rightarrow K_s^p$   
**universe**  $U$ , **free** univ.  $F_s = s$ , **cofree** univ.  $K^p = p$ .  $F_s^p = K_s^p$
4. THEOREM: Every map is homomorphic and continuous
5. TAC = **complete dense extension** of a pretac.
6. ARITY = functorial sorts ( $s^2$ ) & properties (algebra, coalgebra, ...)

## 1a. PROGRAM: Algebra

Translated into algebraic language, the Yoneda lemma treats multi-sorted unary algebras.

That is, an algebra  $(U_s, U_t, \dots, f, g, \dots)$  consists of a family  $\langle U_s \rangle_{s \in S}$  of sets indexed by sorts  $s \in S$  and a list of unary operations  $f : U_s \rightarrow U_t$  between those sets.

Up to isomorphism there is one free algebra generated by a variable  $x_s$  for each sort  $s$ ; let  $F_s$  denote a specific such.

Yoneda:  $\exists$  bijections  $\alpha_s : U_s \rightarrow \text{Hom}(F_s, U)$  and  $\beta_{st} : \text{Clo}_{st}(U) \rightarrow \text{Hom}(F_t, F_s)$  s.t. for all sorts  $s \in S$ , for all  $a \in U_s$  and all  $f_U : U_s \rightarrow U_t$  in the clone  $\text{Clo}_{st}(U)$ ,  $\alpha_t(f_U(a)) = \alpha_s(a)\beta_{st}(f_U)$ .

That is, the applicative structure internal to  $U$  is dually imitated by the homomorphic structure between  $U$  and the free algebras on one generator.

In this view  $n$ -ary operations  $U^n \rightarrow U$  are mimicked as homomorphisms from  $F_s$  to  $F_{s^n}$  where  $s^n$  is a product sort, treated later.

## 1b. PROGRAM: Topology

Ordinary topology can be understood as dual to algebra, with a single dual sort or **property** *in* giving rise to a cofree algebra  $K^{in}$ , namely the Sierpinski space. The two points of  $K^{in}$ , as the two continuous functions from  $1$  to  $K^{in}$ , are to be understood as the two possible values of the *in* property, making this property a predicate.

An open  $x$  of a space  $S$  is a morphism from  $S$  to  $K^{in}$ . A point  $a : 1 \rightarrow S$  is *in* open  $x$  just when  $xa : 1 \rightarrow K^{in}$  is the point  $1$  of  $K^{in}$ .

Our more general notion of topology provides for a set  $P$  of properties, as well as for dependencies between properties dual to the operations between sorts.

## 2. PRETAC

A **pretopoalgebraic category**  $\mathcal{C} = (C, S, P)$ , or pretac for short, is a bipointed category, meaning an (abstract) category  $C$  with distinguished objects of two kinds forming respective sets  $S$  and  $P$ .

Every category becomes a pretac by specifying  $S$  and  $P$ .

This selection imbues all objects of the category with concrete topoalgebraic structure, including the selected objects, and makes all morphisms concrete maps acting on that structure.

Different selections turn the same object into different universes and the same morphism into different maps.

We will show that every map is homomorphic and continuous with respect to the imputed topoalgebraic structure.

The "pre" indicates that not all possible universes need exist in a pretac, even up to isomorphism. It furthermore indicates that the continuous homomorphisms between two universes that are present need not appear exactly once: they may be absent, or be duplicated.

## 3a. DEFINITIONS: Universes and maps

A choice of sorts and properties elevates each object to the status of universe.

A **universe** is an object of  $C$  in the context of a choice of sorts and properties.

A universe is **free** (**cofree**) when it is isomorphic in  $C$  to a sort (property).

Accordingly we denote sort  $s$  by  $F_s$  and dually property  $p$  by  $K^p$  when treating them as universes.

Universes isomorphic to a sort or property are themselves free or cofree, but will not in general be the canonical such.

A **map** is a morphism of  $C$  in the context of a choice of sorts and properties.

## 3b. DEFINITIONS: Elements and terms

An **element** of a universe  $U$  of sort  $s$  is a map  $a : F_s \rightarrow U$ . We denote the set of elements of  $U$  of sort  $s$  by  $U_s = \text{Hom}(s, U)$ .

A **term** of sort  $s \rightarrow t$  is an element of  $F_s$  of sort  $t$ , that is, a map  $f^* : F_t \rightarrow F_s$ . Each universe  $U$  **interprets** each such term  $f^*$  contravariantly as the function  $f_U : U_s \rightarrow U_t$  mapping each element  $a \in U_s$  (that is,  $a : F_s \rightarrow U$ ) to  $af : F_t \rightarrow U$ , i.e.  $af^* \in U_t$ .

The notation  $f^*$  indicates that the term is to be interpreted contravariantly, namely by right action on elements  $a$ . The right action can be understood as substitution of the term  $f$  in  $a$ .

### 3c. DEFINITIONS: States, dependencies, qualia

A **state** of a universe  $U$  expressing property  $p$  is a map  $x : U \rightarrow K^P$ . We denote the set of states of  $U$  expressing property  $p$  by  $U^P = \text{Hom}(U, p)$ .

A **dependency** of property  $q$  on property  $p$  is a state of  $p$  expressing  $q$ , that is, a morphism  $d : K^P \rightarrow K^q$ . Each universe  $U$  **interprets** each such  $d$  as the function  $d_U : U^P \rightarrow U^q$  mapping each state  $x \in U^P$ , as  $x : U \rightarrow K^P$ , to  $dx : U \rightarrow K^q$ , i.e.  $dx \in U^q$ .

Example: The Sierpinski space  $K^{in}$  has three dependencies on itself, namely its three opens.

Qualia are schizophrenically states of free universes and elements of cofree universes. In particular the sets  $F_S^P$  and  $K_S^P$  are equal.  $K^{in}$  has two elements, of sort *point*.

### 3d. DEFINITIONS: Maps

For each morphism  $h : U \rightarrow V$ ,

- 1 the **left action** of  $h$  on  $U_s$  is the function  $h_s = \text{Hom}(s, h) : U_s \rightarrow V_s$  defined by  $h_s(a) = ha$  for all  $a \in U_s$ .
- 2 the **right action** of  $h$  on  $V^P$  is the function  $h^P = \text{Hom}(h, p) : V^P \rightarrow U^P$  defined by  $h^P(x) = xh$  for all  $x \in V^P$ .
- 3 the **action** of  $h$  is the pair  $(h_s, h^P)$  consisting of the family  $h_s$  of left actions  $h_s : U_s \rightarrow V_s$  and the family  $h^P$  of right actions  $h^P : V^P \rightarrow U^P$ .

A **map** is a morphism interpreted by  $S$  and  $P$  as an action.

Different choices of  $S$  and  $P$  interpret the same morphism as different maps.

## 4. Continuous and homomorphic

A map  $h : U \rightarrow V$  is **homomorphic** when its left action on elements of  $U$  commutes with all operations  $f_U$  of  $U$ .

It is **continuous** when its inverse maps states of  $V$  to states of  $U$  and commutes with all dependencies  $d^U$  of  $U$ . Here the **inverse image** under  $h : U \rightarrow V$  of a state  $x$  of  $V$  is the state  $xh$  of  $U$ .

**Theorem** *Every map is homomorphic and continuous.*

**Proof** The left action of  $h : U \rightarrow V$  on  $f_U(a)$  realized as  $af$  is  $h(af)$ . By associativity this is  $(ha)f$  or  $f_U(h(a))$ . States are trivially preserved by inverse image, and maps commute with dependencies by associativity, dual to the argument for operations.  $\square$

## 5a. Equivalence

The **equational theory** of a pretac consists of the commutative diagrams formed exclusively from operations, dependencies, and qualia. It is immediate that for every universe  $U$ , the operations and dependencies as interpreted in  $U$  satisfy the equations of the theory.

Maps  $h, k : U \rightarrow V$  are called **equivalent** when they have the same action.

**Proposition** *Equivalence is identity for elements and states.*

**Proof** Let  $a, b$  be equivalent elements of  $U_s$ . Then  $a = a1_s = b1_s = b$ . For equivalent states  $x, y$  of  $U^p$ ,  $x = 1_p x = 1_p y = y$ .  $\square$

This applies *a fortiori* to operations and dependencies, being respectively elements and states.

## 5b. Actions are adjoint

Call an action **adjoint** when for every left action  $h_s$  and right action  $h^P$  we have  $xh_s(a) = h^P(x)a$ .

**Proposition** The action of every map is adjoint.

**Proof**  $h_s(a)$  is given by  $ha$  and  $h^P(x)$  by  $xh$ . Hence adjointness holds just when  $x(ha) = (xh)a$ . But this is just associativity.  $\square$

Call a pretac **extensional** when equivalence is identity for all maps.

**Proposition** *Equivalence is a congruence.*

**Corollary** *The quotient of a pretac by equivalence is an extensional pretac.*

We assume henceforth that all pretacs are extensional.

## 5c. Dense extensions

A pretac  $(C', S, P)$  **extends**  $(C, S, P)$  when  $C$  is a subcategory of  $C'$  and all elements and states in  $C'$  of objects of  $C$  are in  $C$ . That is, extension adds no new elements or states to universes of  $C$ .

In particular extension adds no new operations, dependencies, or qualia.

A pretac is **dense** (in the sense of densely embedding  $S$  and  $P$ ) when every proper (nonidentity) extension has a new object.

## 5d. Dense extensions

**Proposition** Every pretac has a dense extension with no new objects.

**Proof** For each non-sort  $U$ , non-property  $V$ , and adjoint action  $U \rightarrow V$  not representing a map  $h : U \rightarrow V$ , adjoin  $h : U \rightarrow V$  with composites with elements of  $U$  and states of  $V$  defined to correspond to that action. Adjointness ensures associativity of  $yha$  for all elements  $a$  of  $U$  and states  $y$  of  $V$ . Define each remaining composite  $hg : T \rightarrow V$  for non-elements  $g : T \rightarrow U$  to be the map represented by the composition of the actions of  $h$  and  $g$ , and likewise for composites  $kh : U \rightarrow W$  for non-states  $k : V \rightarrow W$ .  $\square$

## 5e. Complete pretacs

A dense pretac is **complete** when it is equivalent to its every dense extension.

A **tac** (for TopoAlgebraic Category) is a complete dense pretac.

Density of the rationals in any Archimedean field, and the reals as a complete Archimedean field, are posetal instances of these notions.

## 6a. ARITY: Secondary sorts and properties

To accommodate algebra and coalgebra with other operations than those between two sorts, such as  $f : s^2 \rightarrow s$  and  $f : s \rightarrow s + s$ , we introduce the notions of secondary sort and property, for example  $s^2$ ,  $s + s$ ,  $p^2$ ,  $p + q$ , etc.

A secondary sort may participate in the signature as either the domain or codomain of operations, dependencies, and qualia.

*Examples.* (note contravariance)

- 1 Group multiplication  $m^* : s \rightarrow s^2$
- 2 Projections  $\pi_1^*, \pi_2^* : s \rightarrow s^2$
- 3 Coalgebra  $d^* : A + A \rightarrow A$

## 6b. ARITY: Projections and diagonal

The projections  $\pi_1^*, \dots, \pi_n^* : s \rightarrow s^n$  interpret morphisms  $t : s^n \rightarrow U$  as  $n$ -tuples  $(t\pi_1, \dots, t\pi_n)$  where each  $t\pi_i \in U_s$ .

Inclusions  $i^* : s + t \rightarrow s$ ,  $j^* : s + t \rightarrow t$  work dually, viz.

$\forall a \in U_s. \exists ! ai^* \in U_{s+t}$ , and

$\forall b \in U_t. \exists ! bj^* \in U_{s+t}$ .

## 6c. ARITY: Limits on secondary sorts and properties

In a nonextensional pretac each such  $n$ -tuple of elements of sort  $s$  in a universe  $U$  may appear any number of times in  $\text{Hom}(s^n, U)$ .

In a dense pretac we want each such  $n$ -tuple to appear exactly once.

This can be accomplished neatly by treating secondary sorts and properties as ordinary universes inasmuch as they do not participate in the definitions of element, state, action, extensionality, and extension, and moreover compose only with their associated projections or inclusions, at first.

Extensionality then identifies multiple appearances of an  $n$ -tuple, while density creates all  $n$ -tuples.

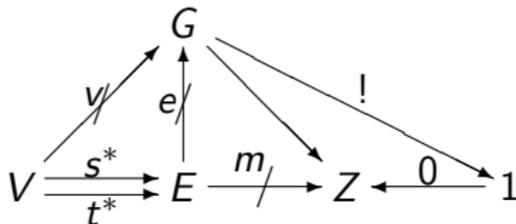
$\text{Hom}(s^n, U)$  is then in a natural bijection with  $U_s^n$ .

This approach dualizes to coproducts, and more generally to limits and colimits.

The remaining composites are then filled in when possible. When not the offending object is discarded (full subcategory).

## 6d. Example

Take the two sorts to be  $V$  and  $E$  as usual for graphs. Take the one primary property to be  $Z$  as the bi-infinite path (a connected acyclic directed graph, unit in-degree and out-degree for all vertices), Take  $1$  to be the 0-ary property  $p^0$ , aka a final object.



where  $mt^* = ms^* + 1$ . The map  $0!$  forces a state labeling edge  $e \in G_V$  with  $0$ , for every  $v \in G_V$ .

Graphs labelable in this way are just the regular graphs, namely those for which every path from vertex  $u$  to vertex  $v$  has the same length.

Call a graph **discrete** when there exist enough states from  $G$  to  $Z$  to label any given vertex or edge with all possible integers.

The discrete graphs in this tac form the category of regular graphs.