Defining Identity

1. The Account

The preferred account of identity, I suggest, involves a Gentzen-style approach to the logical connectives - that is, an approach in which the logical connectives have their meanings specified via introduction and elimination rules. In the case of identity, one very natural choice for the introduction and elimination rules is as follows:

Introduction Rule for Identity $$\frac{Fx}{x = x}$$

Elimination Rule for Identity $$\frac{(x = y),\ Fx}{Fy}$$

2. The Proofs

What now needs to be shown is that, given these introduction and elimination rules, identity, this defined, has the properties it is normally taken to have. What are those properties? Generally, I think, they are taken to be as follows:

Reflexivity: $a = a$

Symmetry: $(a = b) \Rightarrow (b = a)$

Transitivity: $[(a = b)\ \&\ (b = c)] \Rightarrow (a = c)$

Substitutivity: $[(a = b)\ \&\ Fa] \Rightarrow Fb$

Let us now consider how one might attempt to prove these four propositions on the basis of the suggested introduction and elimination rules for identity.

Substitutivity: $[(a = b)\ \&\ Fa] \Rightarrow Fb$

The proof of substitutivity is a quick consequence of the elimination rule for identity:

$(a = b)\ \&\ Fa$	An assumption for a conditional proof
$(a = b)$	Elimination of conjunction.
Fa	Elimination of conjunction.

Now use the following instance of the elimination rule for identity: $$\frac{(a = b),\ Fa}{Fb}$$

Fb

$[(a = b)\ \&\ Fa\ ] \rightarrow Fb$	Discharge of the initial assumption

This conditional proof shows that it is a theorem that $[(a = b)\ \&\ Fa] \rightarrow Fb$. We therefore have that

$[(a = b)\ \&\ Fa] \Rightarrow Fb$.

Symmetry: $\mathbf{(a = b) \Rightarrow (b = a)}$

$a = b$ An assumption for a conditional proof

Now use the following instance of the introduction rule for identity: $\dfrac{a = b}{a = a}$

$a = a$

Now use the following instance of the elimination rule for identity: $\dfrac{(a = b), (a = a)}{b = a}$

Comment: In '($a = a$)', view the first occurrence of 'a' as the subject term, and view the '$= a$' part as the predicate, and similarly for '($b = a$)'.

$b = a$

$(a = b) \rightarrow (b = a)$ Discharge of the initial assumption

It is a theorem, then, that $(a = b) \rightarrow (b = a)$. It is therefore true that

$(a = b) \Rightarrow (b = a)$.

Transitivity: $\mathbf{[(a = b) \;\&\; (b = c)] \Rightarrow (a = c)}$

$(a = b)$ & $(b = c)$ An assumption for a conditional proof

$(b = a)$ & $(b = c)$ Via Symmetry: $(a = b) \Rightarrow (b = a)$

Now use the following instance of the elimination rule for identity: $\dfrac{(b = a), (b = c)}{a = c}$

$a = c$

$[(a = b) \;\&\; (b = c)] \rightarrow (a = c)$ Discharge of the initial assumption

It is a theorem, then, that $[(a = b) \;\&\; (b = c)] \rightarrow (a = c)$. It is therefore true that

$[(a = b) \;\&\; (b = c)] \Rightarrow (a = c)$.

Reflexivity: $\mathbf{a = a}$

Fa An assumption for a conditional proof

$a = a$ Introduction Rule for Identity

$Fa \rightarrow (a = a)$ Discharge of the initial assumption

$\neg Fa$ An assumption for a conditional proof

$a = a$ Introduction Rule for Identity

$\neg Fa \rightarrow (a = a)$ Discharge of the initial assumption

$[Fa \rightarrow (a = a)] \;\&\; [\neg Fa \rightarrow (a = a)]$ Conjunction

Fa v $\neg Fa$ Tautology

$a = a$

3. Is the Relation in Question Identity?

Do the introduction and the elimination rules together suffice to specify a single relation, and if so, is the relation so specified identity?

Identity, as ordinarily understood, certainly satisfies the Introduction and Elimination Rules set out above, since the following two propositions are true:

$Fx \Rightarrow (x = x)$

$[(x = y) \,\&\, Fx] \;\Rightarrow\; Fy$

Suppose now that there is some other relation, associated with some predicate '*Rxy*', that also satisfies both the Introduction Rule and the Elimination Rule. The proofs just set out involved the use of formulas of the form '$x = y$' to represent a dyadic relation satisfying the Introduction Rule and the Elimination Rule set out above, but one could just as well have used '*Rxy*' to represent that dyadic relation. So what those proofs show is that ***any*** relation that satisfies the Introduction Rule and the Elimination Rule must be reflexive, symmetric, transitive, and satisfy substitutivity.

One can now prove that any relations that satisfy both the Introduction and the Elimination Rules are **necessarily coextensive**. First of all, given that '*Rxy*' satisfies substitutivity, we have:

(1) $[Rab \,\&\, Fa] \Rightarrow Fb$

Now let '*Fx*' be '$a = x$', so that '*Fa*' is '$a = a$', and '*Fb*' is '$a = b$'. Then we have:

(2) $[Rab \,\&\, (a = a)] \Rightarrow (a = b)$

But identity is refle*x*ive, so we have:

(3) $\Box(a = a)$

Hence, we have, from (2) and (3):

(4) $Rab \Rightarrow (a = b)$

Similarly, given that identity satisfies substitutivity, we have

(5) $[(a = b) \,\&\, Fa] \Rightarrow Fb$

Now let '*Fx*' be '*Rax*', so that '*Fa*' is '*Raa*' and '*Fb*' is '*Rab*'. Then we have:

(6) $[(a = b) \,\&\, Raa] \Rightarrow Rab$

But in view of the fact that the relation associated with the predicate '*Rxy*' is necessarily refle*x*ive, we have

(7) $\Box Raa$

Hence we have, from (6) and (7):

(8) $(a = b) \Rightarrow Rab$

So in view of (4) and (8), we have that

(9) $[Rab \Rightarrow (a = b)] \,\&\, [(a = b) \Rightarrow Rab]$

There are, of course, cases where two predicates are logically equivalent, but where it might plausibly be claimed that the associated properties, or relations, are distinct - such as in the case of the predicates 'is an equilateral Euclidean triangle' and 'is an equiangular Euclidean triangle'. In the absence, however, of some suggestion of what the relation might be that is both distinct from identity, and satisfies (7), it seems reasonable to conclude that the relation of identity is the only relation that satisfies both the Introduction Rule for Identity and the Elimination Rule for Identity, and thus that those two rules suffice for the definition of identity.