Formulas for the evolute of an ellipse

In its second point, the **ALGORITHM** on the bottom of page 237 suggests that one should re-parametrize the family of lines in the style of Equation (3.38) on page 324 (with the parameter s being the same as the slope). In its third point, that **ALGORITHM** instructs one to express the evolute in the form of Equation (3.40). The link between the two equations is of course the function f, which is, so to speak, discovered in Equation (3.38) and then applied through Equation (3.40).

Students could, I believe, carry this out completely on their own, with enough thought. The re-parametrization step, however, may be a tad unfamiliar. Therefore I am supplying one example.

We will find a parametric expression for the evolute of an ellipse, which we take as parametrized by

$$\langle x_0, y_0\rangle \;=\; \langle \gamma_1(t_0), \gamma_2(t_0)\rangle \;=\; \langle a\cos t_0, b\sin t_0\rangle. \tag{1}$$

(Following the general convention of §3.4, we are taking t_0 to indicate a parameter value along the ellipse; the corresponding point on the ellipse is then denoted $\langle x_0, y_0\rangle$.

In this case, the unnumbered equation on page 237 becomes

$$b\cos t_0(y - y_0) \;=\; a\sin t_0(x - x_0), \tag{2}$$

which can easily be rewritten as

$$\begin{aligned} y \;&=\; \left(\frac{a}{b}\tan t_0\right)x \;-\; \left(\frac{a}{b}\tan t_0\right)x_0 \;+\; y_0 && (3)\\ &=\; \left(\frac{a}{b}\tan t_0\right)x \;-\; \frac{a^2}{b}\sin t_0 \;+\; b\sin t_0 && (4)\\ &=\; \left(\frac{a}{b}\tan t_0\right)x \;+\; \frac{b^2 \;-\; a^2}{b}\sin t_0. && (5)\end{aligned}$$

As we said, we wish to be able to use the solution (3.40) to Clairaut's Equation; first we need to recast Equation (5) in the form of Equation (3.38). Here again is Equation (3.38):

$$y \;=\; sx \;+\; f(s). \tag{6}$$

To reconcile this equation with Equation (5), we need only let

$$s \;=\; \frac{a}{b}\tan t_0. \tag{7}$$

From this it follows almost immediately (by examining the appropriate right triangle) that

$$\sin t_0 \;=\; \frac{sb}{\sqrt{a^2 + s^2b^2}}. \tag{8}$$

Thus Equation (5) is immediately rewritten as

$$y \;=\; sx \;+\; \frac{s(b^2 - a^2)}{\sqrt{a^2 + s^2 b^2}}. \tag{9}$$

In other words, we now have (6) holding, with

$$f(s) \;=\; \frac{s(b^2 - a^2)}{\sqrt{a^2 + s^2 b^2}}. \tag{10}$$

We trust the reader to carry out the necessary symbolic calculations to obtain

$$f'(s) \;=\; \frac{a^2(b^2 - a^2)}{(a^2 + s^2 b^2)^{3/2}}. \tag{11}$$

Now the parametrization of the evolute is immediate from Equation (3.40), which we repeat here:

$$\langle x(u), y(u)\rangle \;=\; \langle -f'(u), -uf'(u) \;+\; f(u)\rangle \tag{12}$$

Substituting formulas for f and f' from Equations (10) and (11), we obtain

$$x(u) \quad = \quad \frac{a^2(a^2 - b^2)}{(a^2 + u^2 b^2)^{3/2}} \tag{13}$$

$$y(u) \quad = \quad -\frac{u^3 b^2(a^2 - b^2)}{(a^2 + u^2 b^2)^{3/2}} \tag{14}$$

From these formulas the reader may easily program a drawing of the evolute by itself.