Technical Appendix to “Nonparametric Estimation of Large Auctions with Risk Averse Bidders”

by Xiaodong Liu

Proof of Lemma A.1. Consider the risk neutral bidding function $s_{RN,n}(\cdot)$ given by the differential equation

$$v - s_{RN,n}(v) = \frac{1}{n - 1} \frac{F(v)}{f(v)} s'_{RN,n}(v).$$

Let $G_{RN,n}(\cdot)$ be the distribution function of the equilibrium bids $b = s_{RN,n}(v)$ and $g_{RN,n}(\cdot)$ be its density. As $G_{RN,n}(b) = F(v)$ and $g_{RN,n}(b) = f(v)/s'_{RN,n}(v)$, the corresponding inverse bidding function is

$$s_{1,RN,n}(b) = b + \frac{1}{n - 1} \int g_{RN,n}(b) db.$$ Guerre, Perrigne and Vuong (2000) have shown $g_{RN,n}(b) > 0$ for all $n$. As $\lim_{n \to \infty} s_{RN,n}(v) = v$ by Proposition 1 in Fibich and Gavious (2010), we have

$$\lim_{n \to \infty} g_{RN,n}(s_{RN,n}(v)) = f(v)$$

where $f(v) > 0$ by the definition of $F_R$. It follows that

$$\lim_{n \to \infty} \sup |s_{RN,n}(b) - b| = 0$$

or, equivalently, $\lim_{n \to \infty} \sup |v - s_{RN,n}(v)| = 0$. As $0 \leq v - s_n(v) \leq v - s_{RN,n}(v)$ for all $v \in S(F)$ (Riley and Samuelson, 1981), the uniform equicontinuity of $s_n(\cdot)$ follows by the uniform convergence of $s_n(v)$ and the compactness of $S(F)$ (Rudin, 1976, Theorem 7.24).

Proof of Lemma A.2. It follows from Proposition 3 that

$$\tilde{G}_n(b) = \frac{1}{nL} \sum_{i,t} 1(B_{it} \leq b) = \frac{1}{nL} \sum_{i,t} 1(G_n(B_{it}) \leq G_n(b)) = \frac{1}{nL} \sum_{i,t} 1(u_{it} \leq G_n(b)),$$

where $u_{it} = G_n(B_{it})$ is uniformly distributed on $[0, 1]$ since $B_{it} \sim G_n(\cdot)$. Hence,

$$|\tilde{G}_n(b) - G_n(b)|_{0,S(G_n)} = \frac{1}{nL} \sum_{i,t} 1(u_{it} \leq G_n(b)) - G_n(b)|_{0,S(G_n)}$$

where the last step holds because the empirical distribution of uniform distribution on $[0, 1]$ (which does not depend on $n$) converges uniformly to the true distribution at the rate of $r_G$ by the Chung-Smirnov theorem (Chung, 1949).

Proof of Lemma A.3. Following the proof of Lemma B.2 in Guerre, Perrigne and Vuong (2000),
the proof is divided into three steps. The first step studies the uniform bias of \( \hat{g}_n (\cdot) \), the second step studies its uniform variance bound, and the last step establishes the exponential-type inequality.

Step 1 (Uniform Bias). Let

\[
\hat{g}_n (b) = \frac{1}{nLh} \sum_{i,t} K\left( \frac{B_i - b}{h} \right) = \int K (u) g_n (b + hu) \, du.
\]

Without loss of generality, suppose \( u \geq 0 \). Then for \( b \in C_n \) and \( L \) sufficiently large, \( b \in [b, b + hu] \subset C'_n \), where \( C'_n \) is a closed inner subset of \( S (G_n) \). Since \( g_n (\cdot) \) admits \( R \) continuous derivatives, a Taylor expansion gives

\[
g_n (b + hu) - g_n (b) \leq h u g_n^{(1)} (b) + \cdots + \frac{h u^{R-1}}{(R-1)!} g_n^{(R-1)} (b) + \frac{|h u|^R}{R!} |g_n|_{R,C'_n}.
\]

As \( K (\cdot) \) is of order \( R \), moments of order strictly smaller than \( R \) vanish. We have

\[
|\operatorname{E}[\hat{g}_n (b)] - g_n (b)|_{0,C_n} = \sup_{b \in C_n} |\int K (u) (g_n (b + hu) - g_n (b)) \, du| \leq h^R M^R |g_n|_{R,C'_n},
\]

where \( M^R = \frac{1}{R!} \int |u|^R K (u) \, du \). It follows from the definition of \( r_g \) and \( h \) that

\[
r_g |\operatorname{E}[\hat{g}_n (b)] - g_n (b)|_{0,C_n} \leq \phi^R M^R |g_n|_{R,C'_n}.
\]

(1)

Step 2 (Uniform Variance). Consider a density \( g_n^* (b) \), such that \( g_n^* (b) = g_n (b) \) if \( b \in S (G_n) \) and \( g_n^* (b) = 0 \) otherwise. For \( b \in C_\infty \), where \( C_\infty = \lim_{n \to \infty} C_n \), \( \operatorname{Var}[\hat{g}_n (b)] = \operatorname{Var}[\frac{1}{nLh} \sum_{i,t} K\left( \frac{B_i - b}{h} \right)] = \frac{1}{nLh^2} \operatorname{Var}[K(\frac{B_i - b}{h})] \leq \frac{1}{nLh^2} \operatorname{E}[K(\frac{B_i - b}{h})]^2 = \frac{1}{nLh} \int K^2 (u) g_n^* (b + hu) \, du \). Let \( Q = \int K^2 (u) \, du \), we have

\[
|\operatorname{Var}[\hat{g}_n (b)]|_{0,C_\infty} \leq \frac{1}{nLh} Q |g_n^*|_{0,C_\infty} \leq \frac{1}{\phi^2 g^2 \log (nL)} Q |g_n|_0.
\]

(2)

Step 3 (Exponential-type Inequality). In this step, we establish the exponential-type inequality for the probability of deviation of \( \hat{g}_n (b) - g_n (b) \) in sup-norm over \( C_n \). Let \( e (c_1, c_2) = c_1 + 2c_2 |K|_1 + \phi^R M^R |g_n|_{R,C'_n}, \) where \( c_1, c_2 \) are strictly positive constants. From the triangular inequality and (1), we have

\[
\Pr \left[ r_g |\hat{g}_n (b) - g_n (b)|_{0,C_n} > e (c_1, c_2) \right] \leq \Pr \left[ r_g |\hat{g}_n (b) - \operatorname{E}[\hat{g}_n (b)]|_{0,C_n} > e (c_1, c_2) - \phi^R M^R |g_n|_{R,C'_n} \right].
\]

(3)

Let \( \hat{g}_n (b) - \operatorname{E}[\hat{g}_n (b)] = (1/nL) \sum_{j=1}^{nL} \zeta_{j,nL} (b) \), where \( \zeta_{j,nL} (b) = \frac{1}{h} K\left( \frac{B_i - b}{h} \right) - \frac{1}{h} \operatorname{E}[K\left( \frac{B_i - b}{h} \right)] \). By the triangular inequality we have

\[
|r_g \zeta_{j,nL}| \leq \frac{2r_g}{h} |K|_0 = \frac{2nL}{\phi r_g \log (nL)} |K|_0.
\]

As the \( \zeta_{j,nL} \)'s are independent zero-mean random variables, it follows from (2) that

\[
\operatorname{Var} \left( r_g \zeta_{j,nL} \right) = nL r_g^2 \operatorname{Var} (\hat{g}_n) \leq \frac{2^2}{\phi^2 g^2 \log (nL)} Q |g_n|_0.
\]
\[
\frac{nL}{\phi \log(nL)} Q |g_n|_0. \quad \text{Hence, the Bernstein inequality gives}
\]
\[
\Pr \left[ r_g |g_n (b) - E[g_n (b)]| > c_1 \right] = \Pr \left[ \left| \sum_{j=1}^{nL} r_g \zeta_{j,nL} (b) \right| > nLC_1 \right] \leq 2 \exp \left[ -\frac{\phi c_1^2 \log(nL)}{2Q |g_n|_0 + 4c_1 |K|_1} \right],
\]
for any \( b \in C_n, c_1, n \) and \( L \).

Note that \( C_n \subset C_\infty \) for all \( n < \infty \) and \( C_\infty = C(V) \). Suppose \( C_\infty \) is covered by \( T \) intervals of the form \( B_t \equiv B(b_t, \Delta) = \{ b \in S(F) : b \in [b_t - \Delta, b_t + \Delta] \} \), where \( b_t \in C_\infty \) and \( \Delta > 0 \). Consider a minimal covering (i.e. a covering with the smallest \( T \)) for \( C_\infty \) with the covering number denoted by \( T(\Delta) \). For any \( b \in B_t \), by the triangular inequality,
\[
r_g |g_n (b) - E[g_n (b)]| \leq \sup_{1 \leq t \leq T(\Delta)} \sup_{b \in B_t} \left| \frac{r_g}{nL} \sum_{j=1}^{nL} (\zeta_{j,nL} (b_t) - \zeta_{j,nL} (b)) \right| + \sup_{1 \leq t \leq T(\Delta)} \left| \frac{r_g}{nL} \sum_{j=1}^{nL} \zeta_{j,nL} (b_t) \right|,
\]
which implies that
\[
\Pr \left[ r_g \sup_{b \in C_n} |g_n (b) - E[g_n (b)]| > e(c_1, c_2) - \phi^R M_R |g_n|_{R,C_n} \right] \\
\leq \Pr \left[ \sup_{1 \leq t \leq T(\Delta)} \sup_{b \in B_t} \left| \frac{r_g}{nL} \sum_{j=1}^{nL} (\zeta_{j,nL} (b_t) - \zeta_{j,nL} (b)) \right| > e(c_1, c_2) - c_1 - \phi^R M_R |g_n|_{R,C_n} \right] \\
+ \Pr \left[ \sup_{1 \leq t \leq T(\Delta)} \left| \frac{r_g}{nL} \sum_{j=1}^{nL} \zeta_{j,nL} (b_t) \right| > c_1 \right].
\]
By the mean value theorem, \( \frac{1}{n} K\left( \frac{b-h}{n} \right) - \frac{1}{n} K\left( \frac{b-h}{n} \right) \leq \frac{\Delta |K|_1}{n^2} \). Therefore, by the triangular inequality, \( |\zeta_{j,nL} (b_t) - \zeta_{j,nL} (b)| \leq \frac{\Delta |K|_1}{n^2} + \frac{\Delta |K|_1}{n^2} = \frac{2\Delta |K|_1}{n} \). Let \( \Delta = c_2 h^2 / r_g \). It follows that
\[
\Pr \left[ \sup_{1 \leq t \leq T(\Delta)} \sup_{b \in B_t} \left| \frac{r_g}{nL} \sum_{j=1}^{nL} (\zeta_{j,nL} (b_t) - \zeta_{j,nL} (b)) \right| \leq \frac{2\Delta |K|_1}{n} = 2c_2 |K|_1 \right].
\]
Hence, by the definition of \( e(c_1, c_2) \),
\[
\Pr \left[ \sup_{1 \leq t \leq T(\Delta)} \sup_{b \in B_t} \left| \frac{r_g}{nL} \sum_{j=1}^{nL} (\zeta_{j,nL} (b_t) - \zeta_{j,nL} (b)) \right| > e(c_1, c_2) - c_1 - \phi^R M_R |g_n|_{R,C_n} \right] \\
= \Pr \left[ \sup_{1 \leq t \leq T(\Delta)} \sup_{b \in B_t} \left| \frac{r_g}{nL} \sum_{j=1}^{nL} (\zeta_{j,nL} (b_t) - \zeta_{j,nL} (b)) \right| > 2c_2 |K|_1 \right] = 0.
\]
Let $P(c_1, c_2) = 2T(c_2h^2/r_g)\exp[-\frac{\delta c_1^2 \log(nL)}{2Q|g_n|_{0,C_n} + 4c_1|K_1|_{0}/(3r_g)}]$. It follows from (3), (5), (6) and (4) that

$$
\Pr[r_g | g_n(b) - g_n(b)|_{0,C_n} > e(c_1,c_2)] \leq \Pr[r_g | g_n(b) - E[g_n(b)]|_{0,C_n} > e(c_1,c_2) - \phi^R M^R |g_n|_{R,C_n}]
$$

$$
\leq \Pr[r_g \sup_{1 \leq t \leq T(\Delta)} |g_n(b_t) - E[g_n(b_t)]| > c_1]
$$

$$
\leq \sum_{t=1}^{T(\Delta)} \Pr[r_g | g_n(b_t) - E[g_n(b_t)]| > c_1]
$$

$$
\leq P(c_1,c_2).
$$

The covering number $T(\Delta)$ is of order $\Delta^{-1}$. Hence $T(c_2h^2/r_g) = O([nL/\log(nL)]^{(R+2)/(2R+1)})$. By taking $c_1$ sufficiently large, $P(c_1,c_2)$ converges as $nL \to \infty$. By Proposition 3, $e(c_1,c_2) = O(1)$.

The desired result follows from the Borel-Cantelli Lemma.

**References**


