V and M diagrams under concentrated and distributed loads

Step 1 – Reactions at the supports

FBD and equilibrium conditions of the entire beam

\[ +\Sigma M_A = 0: \quad B_y(32 \text{ in.}) - (480 \text{ lb})(6 \text{ in.}) - (400 \text{ lb})(22 \text{ in.}) = 0 \]
\[ B_y = +365 \text{ lb} \quad B_y = 365 \text{ lb} \uparrow \]

\[ +\Sigma M_B = 0: \quad (480 \text{ lb})(26 \text{ in.}) + (400 \text{ lb})(10 \text{ in.}) - A(32 \text{ in.}) = 0 \]
\[ A = +515 \text{ lb} \quad A = 515 \text{ lb} \uparrow \]

\[ \Sigma F_x = 0: \quad B_x = 0 \]

Step 2 – Replace the 400lb force by a force-couple system
V and M diagrams under concentrated and distributed loads

**Step 3 – Section AC**
A cut between A and C
FBD and equilibrium equations

\[ + \Sigma F_y = 0: \quad 515 - 40x - V = 0 \]
\[ \Sigma M_A = 0: \quad -515x + 40x(\frac{1}{2}x) + M = 0 \]

\[ V = 515 - 40x \]
\[ M = 515x - 20x^2 \]

\[ 0 < x < 12 \text{ in} \]

**Step 4 – Section CD**
A cut between C and D
FBD and equilibrium equations

\[ + \Sigma F_y = 0: \quad 515 - 480 - V = 0 \]
\[ \Sigma M_C = 0: \quad -515x + 480(x - 6) + M = 0 \]

\[ V = 35 \text{ lb} \]
\[ M = (2880 + 35x) \text{ lb \cdot in.} \]

\[ 12 \text{ in} < x < 18 \text{ in} \]
V and M diagrams under concentrated and distributed loads

Step 5 – Section DB
A cut between D and B
FBD and equilibrium equations

\[ +\uparrow \sum F_y = 0: \quad 515 - 480 - 400 - V = 0 \quad V = -365 \text{ lb} \]
\[ +\uparrow \sum M_3 = 0: \quad -515x + 480(x - 6) - 1600 + 400(x - 18) + M = 0 \]
\[ M = (11,680 - 365x) \text{ lb \cdot in.} \]

18 in < x < 32 in
V and M diagrams under concentrated and distributed loads

Step 6 – Plot V and M diagram

Using the equations for V and M obtained in previous steps

For example: Section AC, $0 < x < 12$ in

$V = 515 - 40x$
$M = 515x - 20x^2$

$x = 0, V = 515$ lb and $M = 0$ lb.in
$x = 12, V = 35$ lb and $M = 3300$ lb.in

For example: Section CD, $12$ in $< x < 18$ in

$V = 35$ lb
$M = (2880 + 35x)$ lb $\cdot$ in.

$x = 12, V = 35$ lb and $M = 3300$ lb.in
$x = 18, V = 35$ lb and $M = 3510$ lb.in
V and M diagrams under concentrated and distributed loads

Note: Why the V and M equations are not valid at the ending points of the section?

For example: Point D, \( x = 18 \) in, there are jumps in V and M diagram

\[ x < 18 \text{ in, } V = 35 \text{ lb and } M = 3510 \text{ lb.in} \]
\[ x > 18 \text{ in, } V = -365 \text{ lb and } M = 5110 \text{ lb.in} \]
V and M diagram on beams with complicated loads

Step 1 – Reactions at the supports

FBD and equilibrium conditions of the entire beam
V and M diagram on beams with complicated loads

Step 2 – Section AB

Free-body diagram for $0 < x < 6 \text{ m.}$

Step 3 – Section BC

Free-body diagram for $6 < x < 12 \text{ m.}$
V and M diagram on beams with complicated loads

Step 4 – Section CD. Take the right part.

Multiple sections between each change of load must be considered in the beam.
Multiple sections between each change of load including reactions must be considered in the beam

Sections AC, CD, DE, ... need to be considered

FBD and equilibrium equations need to be used for each section
Shear force and bending moment diagram
(Distributions of V and M in a beam)

There are two methods to obtain V and diagrams

1. Using FBD and equilibrium equations to determine the equations for V and M as functions of location x (with complete information)

2. Using the relationships among V, M, and $p$ (the distributed load) to determine the important values and trends of V and M diagrams (fast and convenient)

The second method will be discussed.
Relationships among V, M, and \( p \)

Cut a small element from a beam

\( V, M \) and \( p \) are functions of \( x \)

The equilibrium equation in Y direction

\[
V(x) + p(x) \Delta x - V(x + \Delta x) = 0
\]

\[
p(x) = \frac{V(x + \Delta x) - V(x)}{\Delta x}
\]

This means the derivative (the slope) of \( V \) diagram at \( x \) is the value of the applied load
Relationships among V, M, and \( p \)

The moment equilibrium equation about Point O

\[
M(x) - M(x + \Delta x) + V(x) \Delta x + p(x) \Delta x \frac{\Delta x}{2} = 0
\]

\[
V(x) = \frac{dM}{dx}
\]

This means the derivative (the slope) of M diagram at x is the value of V.
Relationships among $V$, $M$, and $p$

Take the second derivative

This means the second derivative (the curvature) of $M$ diagram at $x$ is the value of $p$

These are the relationships among $V$, $M$, and $p$ in the form of derivatives. They can be used to plot $V$ and $M$ diagram.
The relationships among V, M, and p in the form of integration are also useful for plotting V and M diagram.

For V and p,
Integrate on both sides
Integrate from x1 to x2

\[ p \frac{dx}{dx} = \int dv = \int p(x) \, dx \]

\[ V_2 - V_1 = \int_{x_1}^{x_2} p(x) \, dx \quad \rightarrow \quad V_2 = V_1 + \int_{x_1}^{x_2} p(x) \, dx \]

This means the V at x2 is the V at x1 + the area under p curve (from x1 to x2)

Similarly, for V and M
Integrate on both sides from x1 to x2

\[ V(x) = \frac{dM}{dx} \]

\[ M_2 - M_1 = \int_{x_1}^{x_2} V(x) \, dx \quad \quad M_2 = M_1 + \int_{x_1}^{x_2} V(x) \, dx \]

This means the M at x2 is the M at x1 + the area under V curve (from x1 to x2)